

TETRAVALENT EDGE-TRANSITIVE CAYLEY GRAPHS OF FROBENIUS GROUPS

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ABSTRACT. In this paper, we give a characterization for a class of edge-transitive Cayley graphs, and provide methods for constructing Cayley graphs with certain symmetry properties. Also this study leads to construct and characterise a new family of half-transitive graphs.

KEYWORDS. Frobenius group, Edge-transitive graph, Coset graph, Cayley graph

1. INTRODUCTION

Graphs considered in this paper are assumed to be finite, simple, and unless stated otherwise, connected and undirected. For a graph Γ , let $V\Gamma$, $E\Gamma$ and $\text{Aut}\Gamma$ denote its vertex set, edge set and the full automorphism group, respectively. If there exists a subgroup $X \leq \text{Aut}\Gamma$ is transitive on $V\Gamma$ or $E\Gamma$, then the graph Γ is said to be X -vertex transitive or X -edge transitive, respectively. A sequence v_0, v_1, \dots, v_s of vertices of Γ is called an s -arc if $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$, and $\{v_i, v_{i+1}\}$ is an edge for $0 \leq i \leq s-1$. The graph Γ is called (X, s) -arc-transitive, if X is transitive on the s -arcs of Γ ; if in addition X is not transitive on the $(s+1)$ -arcs, then Γ is said to be (X, s) -transitive. In particular, a 1-arc is simply called an arc, and an $(X, 1)$ -arc-transitive graph is called X -arc transitive.

A graph Γ is called a Cayley graph if there exist a group G and a subset $S \subset G \setminus \{1\}$ with $S = S^{-1} := \{g^{-1} \mid g \in S\}$ such that the vertices of Γ may be identified with the elements of G in such a way that x is adjacent to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\text{Cay}(G, S)$. Throughout this paper, denote by $\mathbf{1}$ the vertex of $\text{Cay}(G, S)$ corresponding to the identity of G .

It is well-known that a graph Γ is a Cayley graph of a group G if and only if the full automorphism group $\text{Aut}\Gamma$ contains a subgroup which is regular on vertices and isomorphic to G . In particular, a Cayley graph $\text{Cay}(G, S)$ is vertex-transitive of order $|G|$. However, a Cayley graph is of course not necessarily edge-transitive. Thus, characterizing the Cayley graphs which are edge-transitive is a current hot topic in algebra graph theory. For instance, see [9, 19, 29, 31] for those with valency 4, see [17] for a classification of connected edge-transitive tetravalent Cayley graphs of square-free order, and [5] for a classification of normal edge-transitive Cayley graphs of Frobenius groups of order a product of two primes. In this paper, a characterization is given of tetravalent edge-transitive Cayley graphs of a class of primitive Frobenius groups. This study provides a method for constructing edge-transitive graphs of valency 4, and is then applied to construct a new family of half-transitive graphs. To state this result, we need more definitions.

For an X -vertex-transitive graph Γ and a normal subgroup $N \triangleleft X$, the normal quotient graph Γ_N induced by N is the graph which has vertex set $V\Gamma_N = \{u^N \mid u \in V\Gamma\}$ such that u^N and v^N are adjacent if and only if u is adjacent in Γ to some vertex in v^N . Furthermore, if the valency of Γ_N equals the valency of Γ , then Γ is called a normal cover of Γ_N .

For an integer $m \geq 3$, we denote by $\mathbf{C}_{m[2]}$ the lexicographic product of the empty graph $2\mathbf{K}_1$ of order 2 by a cycle \mathbf{C}_m of size m , which has vertex set $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq 2\}$ such that (i, j) and (i', j') are adjacent if and only if $i - i' \equiv \pm 1 \pmod{m}$.

A group G is said to be a *Frobenius group* if and only if G has the form $G = W:H$ such that each non-identity element of H centralises no non-identity element of W , that is, $xy \neq yx$ for any $x \in W \setminus \{1\}$ and $y \in H \setminus \{1\}$. In particular, G is called a *primitive Frobenius group* if H acts irreducibly on W , refer to [7].

Let \mathbb{F} be a field, R be a group and V be an $\mathbb{F}R$ -module. Suppose that $V = V_1 \oplus \cdots \oplus V_r$ ($r > 1$), where V_i are subspaces of V which are transitively permuted by the action of R . We call R imprimitive on V if there exists such decomposition. Otherwise, R is called primitive on V .

Theorem 1.1. *Let $G = W:H \cong \mathbb{Z}_p^d:\mathbb{Z}_n$ be a primitive Frobenius group, where d, n are integers, and p is a prime. Assume that Γ is a connected tetravalent X -edge-transitive Cayley graph of G , where $G \leq X \leq \text{Aut}\Gamma$. If X is soluble, then one of the following statements holds:*

- (1) G is normal in X , and $X_1 \leq D_8$;
- (2) $G \cong D_{2p}$, $\Gamma \cong \mathbf{C}_{p[2]}$, and $\text{Aut}\Gamma \cong \mathbb{Z}_2^p:D_{2p}$;
- (3) $X = W:((N:H).O)$ with $\text{soc}(X) = W \times L$, and $X_1 = N.O$, where $N \cong \mathbb{Z}_2^l$ with $2 \leq l \leq d$, $L \cong 1$ or \mathbb{Z}_2 , and $O \cong 1$ or \mathbb{Z}_2 , satisfying the following statements:
 - (a) there exist $x_1, \dots, x_d \in W$ and $\tau_1, \tau_2, \dots, \tau_d \in N$ such that $W = \langle x_1, x_2, \dots, x_d \rangle$, $\langle x_i, \tau_i \rangle \cong D_{2p}$ and $N = \langle \tau_i \rangle \times \mathbf{C}_N(x_i)$ for $1 \leq i \leq d$;
 - (b) H does not centralise N , and H is imprimitive on W ;
 - (c) $X/(WN) \cong \mathbb{Z}_n$ or D_{2n} , and Γ is X -arc-transitive if and only if $X/(WN) \cong D_{2n}$;
- (4) $\Gamma_W \cong \mathbf{C}_{\frac{n}{2}[2]}$, Γ is a cover of Γ_W and $X = W:((NH).O)$ such that
 - (i) $X_1 \leq N.O$, $N \cap H \cong \mathbb{Z}_2$, and H normalizes N , but H does not centralise N , where $N \cong \mathbb{Z}_2^l$ with $2 \leq l \leq \frac{n}{2}$, 4 divides n , and $O \cong 1$ or \mathbb{Z}_2 ;
 - (ii) W is the unique minimal normal subgroup of X , and H is imprimitive on W ;
 - (iii) $X/(WN) \cong \mathbb{Z}_{\frac{n}{2}}$ or D_n , and Γ is X -arc-transitive if and only if $X/(WN) \cong D_n$;
- (5) $X = ((WN):H).O$ and Γ is X -arc-transitive if and only if $X/(WN) \cong D_{2n}$, where $W \cong \mathbb{Z}_2^d$, N is a 2-group, and $O \cong 1$ or \mathbb{Z}_2 .

Remarks on Theorem 1.1.

- (a) The Cayley graph Γ in part (1), called a normal edge-transitive graph, is studied in [27]. Furthermore, if $X = \text{Aut}\Gamma$, then Γ is called a normal Cayley graph, introduced in [32].
- (b) H acts irreducibly on W if and only if n does not divide $p^m - 1$ for any proper divisor m of d (such n is called a *primitive divisor* of $p^d - 1$), refer to [6, Proposition 2.3].

- (c) Lemma 4.4 and Lemma 4.5 show that every group X satisfies part (3) or part (4) if and only if H is imprimitive on W , see Constructions 3.3 and 3.5. In addition, H is imprimitive on W if and only if there exists some prime k dividing d such that n divides $k(p^{\frac{d}{k}} - 1)$, see [6, Proposition 2.8].

Theorem 1.2. *Using the notation defined in Theorem 1.1, if X is insoluble, then one of the following holds:*

- (1) $G \cong \mathbb{Z}_p^4 : \mathbb{Z}_5$, $X = W.\overline{X}$ and $\Gamma_W \cong \mathbf{K}_5$, where $\text{soc}(\overline{X}) \cong A_5$, and Γ is constructed as in Construction 3.9;
- (2) $G \cong \mathbb{Z}_p^4 : \mathbb{Z}_{10}$, $X = W.(\overline{X} \times \mathbb{Z}_2)$, and $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$, where $\text{soc}(\overline{X}) \cong A_5$, and Γ is constructed as in Construction 3.11;
- (3) Γ is isomorphic to one of the graphs listed in Table 1.

TABLE 1: Graphs which are not normal edge-transitive.

$\text{Aut}\Gamma$	G	$(\text{Aut}\Gamma)_1$	Γ
$\text{PSL}(3, 3) : \mathbb{Z}_2$	D_{26}	$\mathbb{Z}_3^2 : \text{GL}(2, 3)$	Example 3.13
$\text{PGL}(2, 7)$	D_{14}	S_4	Example 3.13
$\text{PGL}(2, 7)$	$\mathbb{Z}_7 : \mathbb{Z}_3$	D_{16}	Example 3.14
$\text{PGL}(2, 7)$	$\mathbb{Z}_7 : \mathbb{Z}_6$	D_8	Example 3.15
$\text{PSL}(2, 23)$	$\mathbb{Z}_{23} : \mathbb{Z}_{11}$	S_4	Example 3.16
$\text{PGL}(2, 11)$	$\mathbb{Z}_{11} : \mathbb{Z}_5$	S_4	Example 3.16
$\text{PGL}(2, 11) \times \mathbb{Z}_2$	$\mathbb{Z}_{11} : \mathbb{Z}_{10}$	S_4	Example 3.17

A graph is said to be half-transitive if its automorphism group acts transitively on the vertex set and edge set but intransitively on the arc set. Constructing and characterising half-transitive graphs was initiated by Tutte (1965), and is a currently active topic, see [18, 21, 22, 23] for references. Theorem 1.1 provides a method for characterising some classes of half-transitive graphs of valency 4. The following theorem is such an example.

Theorem 1.3. *Let $G = W : \langle h \rangle \cong \mathbb{Z}_p^d : \mathbb{Z}_n$ be a primitive Frobenius group, where $d > 1$ is odd, p is an odd prime, and n is an integer. Let Γ be a connected tetravalent edge-transitive Cayley graph of G . Assume that $\langle h \rangle$ is primitive on W . Then $\text{Aut}\Gamma = G : \mathbb{Z}_2$, Γ is half-transitive, and $\Gamma \cong \Gamma_i = \text{Cay}(G, S_i)$, where $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$, $(n, i) = 1$, and*

$$S_i = \{ah^i, a^{-1}h^i, (ah^i)^{-1}, (a^{-1}h^i)^{-1}\}, \text{ where } a \in W \setminus \{1\}.$$

Moreover, if $p^r i \equiv \pm j \pmod{n}$ for some $r \geq 0$, then $\Gamma_i \cong \Gamma_j$.

2. PRELIMINARY RESULTS

In this section, we quote some preliminary results, which will be used in the subsequent sections.

For a core-free subgroup H of X and an element $g \in X \setminus H$, let $[X : H] := \{Hx \mid x \in X\}$, and define the coset graph

$$\Gamma = \text{Cos}(X, H, H\{g, g^{-1}\}H)$$

with vertex set $[X : H]$ such that Hx and Hy are adjacent whenever $yx^{-1} \in H\{g, g^{-1}\}H$. Then Γ is well-defined, and X induces a subgroup of $\text{Aut}\Gamma$ acting on $[X : H]$ by right

multiplication, namely, $\alpha : Hx \rightarrow Hxa$ for $x, a \in X$. Label v, w the two vertices of Γ corresponding to H and Hg , respectively. Then we have the following lemma.

Lemma 2.1. *For a coset graph $\Gamma = \text{Cos}(X, H, H\{g, g^{-1}\}H)$, we have*

- (a) $\Gamma(v) = \{Hgh|h \in H\} \cup \{Hg^{-1}h|h \in H\}$;
- (b) Γ is X -edge-transitive and X is transitive on the vertices of Γ ;
- (c) Γ is connected if and only if $X = \langle H, g \rangle$;
- (d) $H \cap H^g = X_{vw}$, the stabilizer of the arc (v, w) , where H^g is the conjugate of H by g ;
- (e) the valency of Γ equals

$$\text{val}(\Gamma) = \begin{cases} |H:H \cap H^g| & \text{if } HgH = Hg^{-1}H, \\ 2|H:H \cap H^g| & \text{otherwise;} \end{cases}$$

- (f) Γ is X -arc-transitive if and only if $HgH = Hg^{-1}H$, which yields that $HgH = HoH$ for some (2-element) $o \in \mathbf{N}_X(X_{vw}) \setminus H$ with $o^2 \in X_{vw}$ (refer to [17]). (An element o in the group X is a 2-element if its order is a power of 2).

Moreover, for any X -edge-transitive graph Σ , if X is transitive on $V\Sigma$, then the map $u^x \rightarrow Hx$ with $x \in X$ gives an isomorphism from Σ to $\text{Cos}(X, H, H\{g, g^{-1}\}H)$, where $u \in V\Sigma$, $H = X_u$ and $g \in X \setminus H$ with $u^g \in \Gamma(u)$.

The vertex stabilizer for s -arc-transitive graphs of valency 4 is known (refer to [30]).

Lemma 2.2. *Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected (X, s) -transitive graph of valency 4. Then s and the stabilizer X_1 are listed in the following table,*

s	2	3	4	7
X_1	A_4, S_4	$\mathbb{Z}_3 \times A_4, (\mathbb{Z}_3 \times A_4).\mathbb{Z}_2, S_3 \times S_4$	$\mathbb{Z}_3^2:\text{GL}(2, 3)$	$[3^5]:\text{GL}(2, 3)$

where $[3^5]$ is a 3-group of order 3^5 .

Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected graph. Assume that $X \leq \text{Aut}\Gamma$ is transitive on both $V\Gamma$ and $E\Gamma$. Then we have an important conclusion in the next lemma.

Lemma 2.3. *Let $N \triangleleft X$. If Γ is of valency 4 and X/N is insoluble, then Γ is a normal N -cover of Γ_N .*

Proof. Pick any vertex $u \in V\Gamma$. Let B be an orbit of N acting on $V\Gamma$, which contains u . By Lemma 2.2, the stabilizer X_u is a $\{2, 3\}$ -group. In particular, X_u is soluble. Let K be the kernel of X acting on Γ_N . Then $K_u \leq X_u$, so K_u is soluble. Since N is transitive on B , we have $K = NK_u$. Note that $K/N \cong NK_u/N \cong K_u/(N \cap K_u)$, K/N is soluble. Then $X/K \cong (X/N)/(K/N)$ is insoluble because X/N is insoluble. So $\text{Aut}\Gamma_N$ is also insoluble, hence Γ_N is not a cycle. Since Γ is connected and the valency of Γ_N is a divisor of the valency of Γ , we conclude that Γ_N is of valency 4, and the lemma holds. \square

For a normal edge-transitive Cayley graph $\Gamma = \text{Cay}(G, S)$, let $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$, we have a simple lemma.

Lemma 2.4. *Let $G = W:\langle h \rangle \cong \mathbb{Z}_p^d:\mathbb{Z}_n$ be a primitive Frobenius group, where d, n are integers, and p is a prime. Let $\Gamma = \text{Cay}(G, S)$ be connected of valency 4. Assume that $\text{Aut}\Gamma$ has a subgroup X such that Γ is X -edge-transitive and $G \leq X$. Then $X_1 \leq D_8$.*

Proof. Since Γ is connected, we have $\langle S \rangle = G$, and so $\text{Aut}(G, S)$ acts faithfully on S . Hence $\text{Aut}(G, S) \leq S_4$. By [10, Lemma 2.1], we obtain $X \leq \mathbf{N}_{\text{Aut}\Gamma}(G) = G:\text{Aut}(G, S)$. So $X_1 \leq \text{Aut}(G, S) \leq S_4$. Suppose that 3 divides $|X_1|$. Then X_1 is 2-transitive on S . Hence Γ is $(X, 2)$ -arc-transitive, and all elements in S are involutions, see for example [15].

Pick any $s \in S$. Write $s = \sigma h^i$ where $\sigma \in W$ and i is an integer. Recall that s is an involution, we obtain that $h^{2i} = 1$. By [8, Proposition 12.10], $\text{Aut}(G) \cong \mathbb{Z}_p^d:\text{GL}(1, p^d)$. For a finite group T , it is known that the action of $\text{Aut}(T)$ on $T/\mathbf{Z}(T)$ is permutationally isomorphic to the conjugation action of $\text{Aut}(T)$ on $\text{Inn}(T)$. Since $G \cong \text{Inn}(G)$, it follows from the above fact that we may identify G with $\text{Inn}(G)$ a normal subgroup of $\text{Aut}(G)$. Then write $\text{Aut}(G) = W:M.L$, where $M \cong \mathbb{Z}_{p^{d-1}}$, and $L \cong \mathbb{Z}_d$. Without loss of generality, we may assume that h belongs to M , refer to [6, Proposition 2.5]. For that case, $(h^i)^\eta = h^i$ for any $\eta \in M.L$. Take any $\theta \in \text{Aut}(G)$, θ has the form xyz , where $x \in W$, $y \in M$, and $z \in L$. By easy calculations, we have $s^\theta = \bar{s}h^i$, where $\bar{s} \in W$. It follows that for each $a \in S$, a has the form $\bar{a}h^i$ with $\bar{a} \in W$ because X_1 is transitive on S . Recall that $\langle S \rangle = G$, we have $h = (\sigma_1 h^i)(\sigma_2 h^i) \cdots (\sigma_m h^i)$ where $\sigma_j \in W$ for each j . Since $W \trianglelefteq G$ and $h \neq 1$, we obtain $h = h^i$. Consequently, $\langle h \rangle \cong \mathbb{Z}_2$. By the definition of G , we have $G \cong D_{2p}$, and thus $\text{Aut}(G) \cong \mathbb{Z}_p:\mathbb{Z}_{p-1}$. However, since X_1 is 2-transitive on S , we conclude that $X_1 \cong A_4$ or S_4 , which is impossible. Therefore, $X_1 \leq D_8$. \square

Finally, we quote a result about simple groups, which will be used later.

Lemma 2.5. (Kazarin [12]) *Let T be a non-abelian simple group which has a 2'-Hall subgroup. Then $T = \text{PSL}(2, p)$, where $p = 2^e - 1$ is a prime. Furthermore, $T = GH$, where $G = \mathbb{Z}_p:\mathbb{Z}_{\frac{p-1}{2}}$ and $H = D_{p+1} = D_{2^e}$.*

3. EXISTENCE OF GRAPHS SATISFYING THEOREM 1.1 AND THEOREM 1.2

In this section, we first construct some examples of graphs satisfying Theorem 1.1.

The following construction produces normal edge-transitive Cayley graphs admitting a group X satisfying part (1) of Theorem 1.1.

Construction 3.1. Let $p \geq 5$ be a prime such that p is a primitive divisor of $2^{p-1} - 1$. Let $G = W:\langle h \rangle \cong \mathbb{Z}_2^{p-1}:\mathbb{Z}_p$ be a Frobenius group. By [8, Proposition 12.10], we have $\text{Aut}(G) \cong \mathbb{Z}_2^{p-1}:\text{GL}(1, 2^{p-1}).\mathbb{Z}_{p-1}$, where \mathbb{Z}_{p-1} is the group of Frobenius automorphisms. Arguing similarly as Lemma 2.4, we may write $\text{Aut}(G) = W:M:L$, and h belongs to M , where $M \cong \mathbb{Z}_{2^{p-1}-1}$, and $L \cong \mathbb{Z}_{p-1}$. For this case, we may identify W with a field $\mathbb{F} = \mathbb{F}_{2^{p-1}}$ of order 2^{p-1} and there exists $\alpha \in \mathbb{F}$ of order p such that $\langle h \rangle$ acts on each $x \in W$ by $h : x = \alpha x$. By the definition, G is a primitive Frobenius group.

Let $\mathbb{F}^\# = \langle \omega \rangle$, and let σ be a Frobenius automorphism of order 2. Then $\omega^\sigma = \omega^{2^{\frac{p-1}{2}}}$. Let $X = G:\langle \sigma \rangle$, and let $g = \omega^{2^{\frac{p-1}{2}}+1}h$. Set

$$\Gamma(2, p-1, p) = \text{Cos}(X, \langle \sigma \rangle, \langle \sigma \rangle \{g, g^{-1}\} \langle \sigma \rangle).$$

Lemma 3.2. *Let $\Gamma = \Gamma(2, p-1, p)$ be a graph constructed in Construction 3.1. Then Γ is a connected normal X -edge-transitive Cayley graph of G of valency 4.*

Proof. By the definition, $\langle \sigma \rangle$ is core-free in X , and hence $X \leq \text{Aut}\Gamma$. Now $X = G\langle \sigma \rangle$ and $G \cap \langle \sigma \rangle = 1$, and thus G is regular on the vertex set $[X:\langle \sigma \rangle]$. So Γ is a Cayley graph of G , which has order $2^{p-1}p$.

Let $Y = \langle g, \sigma \rangle$. Noting that $p \geq 5$, we conclude that $2^{\frac{p-1}{2}} \not\equiv -1 \pmod{2^{p-1} - 1}$. It implies that $\omega^\sigma \neq \omega^{-1}$. However, since $\alpha^\sigma = \alpha^{-1}$, we have that $h^\sigma = h^{-1}$. Furthermore, σ induces an automorphism of G . Then we have

$$g^\sigma = (\omega^{2^{\frac{p-1}{2}}+1}h)^\sigma = (\omega^\sigma)^{2^{\frac{p-1}{2}}+1}h^\sigma = \omega^{2^{\frac{p-1}{2}}+1}h^{-1}.$$

Let $\bar{g} = g^\sigma g$. Then $\bar{g} = \omega^{2(2^{\frac{p-1}{2}}+1)}\alpha$. Denote by ℓ the integer $2^{\frac{p-1}{2}} - 1$. Recall that p is a primitive divisor of $2^{p-1} - 1$, we conclude that $(p, \ell) = 1$. Thus $\bar{g}^\ell = \alpha^\ell$ belongs to Y . So does α . Consequently, $\omega^{2(2^{\frac{p-1}{2}}+1)}$ belongs to Y , and so h belongs to Y . Since $\langle h \rangle$ acts irreducibly on W , we obtain that $X = Y$. Thus Γ is connected. It is straightforward to show that $\langle \sigma \rangle \cap \langle \sigma \rangle^g = 1$, and hence $\langle \sigma \rangle \cap \langle \sigma \rangle^g$ has index 2 in $\langle \sigma \rangle$. Since $X \leq \text{Aut} \Gamma$, it follows that Γ is not a cycle. By Lemma 2.1, Γ is connected, X -edge-transitive and of valency 4. \square

Remark. In fact, the graphs in Construction 3.1 really exist. For example, $p = 5, 11, 13, 19$, and so on.

The following construction produces edge-transitive graphs admitting a group X satisfying part (3) of Theorem 1.1 with $L \cong \mathbb{Z}_2$, and $O = 1$.

Construction 3.3. Let $X = W:(N:\langle h \rangle) \cong \mathbb{Z}_p^d : (\mathbb{Z}_2^d : \mathbb{Z}_n)$, where $p = 2^\ell m + 1$ be an odd prime, $m \geq 3$ is an odd number, and $\ell \geq 1$, such that $W \cong \mathbb{Z}_p^d$, $N \cong \mathbb{Z}_2^d$, and $\langle h \rangle \cong \mathbb{Z}_n$ satisfy

- (a) $d > 1$, d divides m , and $2md$ is a primitive divisor of $p^d - 1$;
- (b) $W = \prod_{i=1}^d \langle x_i \rangle$, where $x_i = (1, \dots, 1, x, 1, \dots, 1)$ with $o(x) = p$ for each i ;
- (c) $N = \prod_{i=1}^d \langle \tau_i \rangle$, where $\tau_i = (1, \dots, 1, \tau, 1, \dots, 1)$ with $x^\tau = x^r$ and $r^{p-1} \equiv 1 \pmod{p}$ for each i ;
- (d) $h = c_1 \tau_1^{\frac{p-1}{2m}} (12 \dots d)$, where $c_1 = (c, 1, \dots, 1)$ with $x^c = x$, $\tau^x = \tau$, and $o(c) = 2$.

Let $y = (x_1 h)^{-1}$. Set

$$\Gamma(p, 2, n) = \text{Cos}(X, N, N\{y, y^{-1}\}N).$$

Lemma 3.4. Let $\Gamma = \Gamma(p, 2, n)$ be a graph constructed in Construction 3.3, and let $G = W:\langle h \rangle$. Then Γ is a connected tetravalent X -edge-transitive Cayley graph of Frobenius group G , and G is not normal in X .

Proof. By the definition, N is core-free in X , and hence $X \leq \text{Aut} \Gamma$. Now $X = GN$ and $G \cap N = 1$, and thus G acts regularly on the vertex set $[X:N]$. So Γ is a Cayley graph of G . Obviously, G is not normal in X .

Let $H = \langle h \rangle$. Suppose that $C := \mathbf{C}_H(W) \neq 1$. Then $C = \langle h^\ell \rangle$, where ℓ divides $2md$. Write $\ell = l_1 d + l$, where $0 \leq l_1 < 2m$, and $0 \leq l < d$. Let $\bar{\tau} = \tau_1 \tau_2 \dots \tau_d$, and $\bar{c} = c_1 \dots c_d$, where $c_i = (1, \dots, 1, c, 1, \dots, 1)$ for each i . If $l = 0$, then $h^\ell = \bar{c} \bar{\tau}^{\frac{p-1}{2m} l_1}$ and so $x_1^{h^\ell} \neq x_1$, a contradiction. Thus $l \neq 0$. Then $h^\ell = \bar{c} \bar{\tau}^{\frac{p-1}{2m} l_1} h^l$. Let $(l, d) = k$. Let $k' = l/k$, and $d' = d/k$. Relabeling if necessary, we may rewrite $\{1, \dots, d\} = \{1_1, \dots, i_j, \dots, k_{d'}\}$. Without loss of generality, we may assume that $h^l = h_1 \dots h_k$, where $h_i = (c_{i_1} \tau_{i_1}^{\frac{p-1}{2m}}) (c_{i_2} \tau_{i_2}^{\frac{p-1}{2m}}) \dots (c_{i_{k'}} \tau_{i_{k'}}^{\frac{p-1}{2m}}) (i_1 i_2 \dots i_{d'})$. Then $x_{1_1}^{h^\ell} = x_{1_2}^{r^{\frac{p-1}{2m}(l_1+1)}} \neq x_{1_1}$ because $1_1 \neq 1_2$, a contradiction. Thus H acts faithfully on W .

We claim that H is fixed-point-free on W . Let $U = \langle w \mid w^h = w, w \in W \rangle$. If otherwise, then U is a proper subgroup of W . By Maschke's Theorem, V can be decomposed as $W = U \times V$ such that H normalises both U and V . By the definition of U , H is fixed-point-free on V . Let $k = \dim(V)$. Then $k < d$. By the above paragraph, we conclude that $2md$ divides $p^k - 1$, contrary to our assumption. This establishes the claim. So G is a primitive Frobenius group.

For y defined in Construction 3.3, let $z = y^{-1}y^{\frac{p-1}{2}} = x_1^2$. Then x_1 belongs to $\langle N, y \rangle$. So does x_i for $1 \leq i \leq d$. It follows that $\langle N, y \rangle = X$. Thus Γ is connected. It furthermore implies that $\langle \bar{c} \rangle$ belongs to X , and so $\text{soc}(X) = W \times \langle \bar{c} \rangle$.

Let $\sigma_i = \tau_i^{\frac{p-1}{2}}$ where $1 \leq i \leq d$. Finally, as $\sigma_i^y = \sigma_{i-1}$ for $3 \leq i \leq d$, $\sigma_1^y = \sigma_d$ and $\sigma_2^y = x_1^2 \sigma_1$, we obtain that $N \cap N^y = \langle \sigma_2, \sigma_3, \dots, \sigma_d \rangle \cong \mathbb{Z}_2^{d-1}$. That is to say, $N \cap N^y$ has index 2 in N . Since $X \leq \text{Aut} \Gamma$, Γ is not a cycle. By Lemma 2.1, Γ is connected, X -edge-transitive and of valency 4. \square

Remark. The normal quotient Γ_W induced by W is a cycle (see Lemmas 4.4 and 4.5).

As a matter of fact, there are several groups which are primitive Frobenius groups and satisfy Construction 3.3. For example, $G = \mathbb{Z}_7^3 : \mathbb{Z}_{18}$, $\mathbb{Z}_{13}^3 : \mathbb{Z}_{18}$, $\mathbb{Z}_{41}^5 : \mathbb{Z}_{50}$, and so on.

The following construction produces edge-transitive graphs admitting a group X satisfying part (4) of Theorem 1.1 with $O = 1$.

Construction 3.5. Using the notation in Construction 3.3. Assume $\ell \geq 2$. Let $N = \prod_{i \neq 3} \langle \tau_i^{\frac{p-1}{2}} \rangle \cong \mathbb{Z}_2^{d-1}$, and $h = \tau_1^{\frac{p-1}{4m}} (12 \cdots d)$. Let $X = W : \langle N, h \rangle$, and let $G = W : \langle h \rangle$. Set $y = (x_2 h)^{-1}$, and

$$\Gamma(p, n) = \text{Cos}(X, N, N\{y, y^{-1}\}N).$$

Lemma 3.6. Let $\Gamma = \Gamma(p, n)$ be a graph constructed in Construction 3.5. Then Γ is a connected tetravalent X -edge-transitive Cayley graph of Frobenius group G , and G is not normal in X .

Proof. Obviously, G is not normal in X . Let $\sigma_i = \tau_i^{\frac{p-1}{2}}$ where $1 \leq i \leq d$. By easy calculations, $\sigma_1^y = \sigma_d$, $\sigma_2^y = \sigma_1$, and $\sigma_i^y = \sigma_{i-1}$ for $4 \leq i \leq d$. It follows that σ_3 belongs to $\langle N, y \rangle$, and $N \cap N^y = \langle \sigma_1, \sigma_4, \dots, \sigma_d \rangle \cong \mathbb{Z}_2^{d-2}$. At the same time, we obtain $\sigma_3^y = x_2^2 \sigma_2$, and hence x_2 belongs to $\langle N, y \rangle$. So does x_i for each i . It implies that $\langle N, y \rangle = X$. Consequently, Γ is connected. Arguing similarly as Lemma 3.4, we can obtain that G is a Frobenius group, and Γ is X -edge-transitive Cayley graph of G and of valency 4, the statement follows. \square

Remark. Clearly, $\langle h \rangle$ does not normalise N . In other words, X can't satisfy the properties in part (a) of Lemma 4.4. However, h normalises $\langle N, h^{\frac{n}{2}} \rangle$, namely, X satisfies the properties in part (ii) of Lemma 4.5. Thus $\Gamma_W \cong \mathbf{C}_{\frac{n}{2}[2]}$, where N , W , and Γ appear in Construction 3.5, (see Lemma 4.4 and Lemma 4.5).

The following construction produces edge-transitive graphs admitting a group X satisfying part (5) of Theorem 1.1 with $O = 1$.

Let $n = 3p_1^{l_1} \cdots p_s^{l_s}$ be an odd number, where $3, p_1, \dots, p_s$ are pairwise distinct primes, and $l_i \geq 1$ for each i . Let $G_1 = W_1 : H_1 \cong \mathbb{Z}_2^d : \mathbb{Z}_n$ be a primitive Frobenius group. Let

G_2 be a subgroup of G_1 such that $G_2 = W_2:H_2 \cong A_4$. Write $H_1 = \langle h_1 \rangle$, and $H_2 = \langle h_2 \rangle$ where $h_2 = h_1^{\frac{n}{3}}$.

Let $H = \langle h \rangle$ where $h = (h_1, h_2)$. Let $V = W_1 \times W_2$, and $W = \{(w, 1) \mid w \in W_1\}$. Set

$$G = W:H \text{ and } X = V:H.$$

By the definition, it is easy to show that G is a primitive Frobenius group.

Construction 3.7. Let $R = \langle (w, w), (w, w)^h \rangle$, where $1 \neq w \in W_2$. Set

$$\Gamma(2, d, n) = \text{Cos}(X, R, R\{h, h^{-1}\}R).$$

Lemma 3.8. *Let $\Gamma = \Gamma(2, d, n)$ be a graph constructed in Construction 3.7. Then Γ is a connected tetravalent X -edge-transitive Cayley graph of G , and G is not normal in X . In particular, Γ_W is a cycle.*

Proof. We first prove that R is core-free in X . By the definition of R , we have $R \cong \mathbb{Z}_2^2$. Assume $K \leq R$, and $1 \neq K \trianglelefteq X$. Then $\text{Aut}(K)$ is isomorphic to a subgroup of S_3 . So we conclude that $\mathbf{C}_H(K) \neq 1$, which contradicts the fact that G_1 is a Frobenius group. Thus R is core-free in X . We observe that $R \cap G = 1$, it follows that $|X| = |R||G|$, and so $X = RG$. It implies that G is regular on the vertex set $[X:R]$, and hence Γ is a Cayley graph of G .

By the definition of G_1 , we conclude that $w^{h^3} \neq w$ for any $1 \neq w \in W_1$. It implies that $(w, w)^{h^3}(w, w) \neq 1$, namely, $(w^{h^3}w, 1) \neq 1$. Since H is irreducible on W_1 , implying that W_1 belongs to $\langle R, h \rangle$. So does V . Thus Γ is connected. Arguing similarly as above, G is not normal in X . Clearly, $R \cap R^h = \langle (w, w)^h \rangle \cong \mathbb{Z}_2$. It follows that $R \cap R^h$ has index 2 in R . As $X \leq \text{Aut}\Gamma$, Γ is not a cycle, so that by Lemma 2.1, we obtain Γ is X -edge-transitive and of valency 4. Note that Γ_W is a Cayley graph. By [1, Theorem 1.2], we conclude that Γ_W is a cycle. \square

By Constructions 3.1-3.5, each case of Theorem 1.1 occurs.

We now construct some examples of graphs appearing in Theorem 1.2.

Based on several previous known results, arc-transitive elementary abelian covers of the complete graph \mathbf{K}_5 were classified by Boštjan Kuzman [3]. However, for the completeness, we present here a distinct and independent construction.

Let p be a prime such that 5 is a primitive divisor of $p^4 - 1$. Set

$$V = \langle e_1 \rangle \times \cdots \times \langle e_5 \rangle \cong \mathbb{Z}_p^5.$$

We define an action of A_5 on V as follows:

$$(\prod_{i=1}^5 e_i^{\lambda_i})^g = \prod_{i=1}^5 e_{ig^{-1}}^{\lambda_i}, \text{ where } g \in A_5, \text{ and } 0 \leq \lambda_i \leq 4 \text{ for each } i.$$

By this definition, A_5 acts naturally on V . Let $\bar{e}_i = e_5 e_i^{-1}$ for $1 \leq i \leq 4$. Set

$$W = \langle \bar{e}_1 \rangle \times \langle \bar{e}_2 \rangle \times \langle \bar{e}_3 \rangle \times \langle \bar{e}_4 \rangle.$$

It is straightforward to show that A_5 acts faithfully on W .

Construction 3.9. Let $G = W:\langle h \rangle$ with $h = (12345)$, and let $X = W:N = \mathbb{Z}_p^4:A_5$. Let $R = \text{Alt}\{2, 3, 4, 5\} \cong A_4$, and let $g = \bar{e}_1(15)(24)$. Set

$$\Gamma(p, 4, 5) = \text{Cos}(X, R, RgR).$$

Lemma 3.10. *Let $\Gamma = \Gamma(p, 4, 5)$ be a graph constructed in Construction 3.9. Then Γ is a connected tetravalent $(X, 2)$ -arc-transitive Cayley graph of Frobenius group G , and G is not normal in X . In particular, Γ is a cover of Γ_W , and $\Gamma_W \cong \mathbf{K}_5$.*

Proof. Let $H = \langle h \rangle$. By definition of W , we conclude H is fixed-point-free on W . Since 5 is a primitive divisor of $p^4 - 1$, H acts irreducibly on W . That is to say, G is a primitive Frobenius group. Clearly, N has a decomposition HR . It implies that R is core-free in X , and hence $X \leq \text{Aut} \Gamma$. Now $X = GR$ and $G \cap R = 1$, and so G is regular on the vertex set $[X:R]$. Thus Γ is a Cayley graph of G . Obviously, G is not normal in X .

Denote by u and v the vertices R and Rg , respectively. Then $X_u = R$ and $X_v = R^g$. Let $r = (234)$. Since $X_{uv} = X_u \cap X_v$, a small calculations show $X_{uv} = \langle r \rangle$. By Lemma 2.1, Γ is of valency 4. It is clear that g has order 2, and $r^g = r^{-1}$. So $g \in \mathbf{N}_X(X_{uv})$. Let $\bar{R} = \langle R, g \rangle$. Since $(15)(24) = (12345)(25)(34)$, we conclude that $\bar{e}_1 h$ belongs to \bar{R} . Let $a = (25)(34)$ and $b = (23)(45)$. By easy calculations, we obtain

$$\bar{e}_1 h (\bar{e}_1 h)^a = \bar{e}_1 \bar{e}_2 \bar{e}_3^{-1}, (\bar{e}_1 \bar{e}_2 \bar{e}_3^{-1})^b = \bar{e}_1 \bar{e}_2^{-1} \bar{e}_3 \bar{e}_4^{-1}, \text{ and } (\bar{e}_1 \bar{e}_2^{-1} \bar{e}_3 \bar{e}_4^{-1})^{ab} = \bar{e}_1 \bar{e}_2^{-1} \bar{e}_4^{-1}.$$

Combining the above three equations, we conclude that \bar{e}_3 belongs to \bar{R} . So does \bar{e}_i for $i = 1, 2, 4$. Consequently, $W \leq \bar{R}$. Recall that $\bar{e}_1 h$ is inside in \bar{R} , it follows that h belongs to \bar{R} , forcing $\langle R, g \rangle = X$. Thus Γ is connected. Since X/W is insoluble, by Lemma 2.3, Γ is a cover of Γ_W . Clearly, Γ_W is a Cayley graph of G/W . By [1, Theorem 1.2], we obtain $\Gamma_W \cong \mathbf{K}_5$. \square

Let p be a prime such that 10 is a primitive divisor of $p^4 - 1$. Let

$$V_1 = \langle e_1 \rangle \times \cdots \times \langle e_5 \rangle, \text{ and } V_2 = \langle e_{1'} \rangle \times \cdots \times \langle e_{5'} \rangle$$

such that $V_1 \cong V_2 \cong \mathbb{Z}_p^5$. Set $T = \langle (12345)(1'2'3'4'5'), (12)(1'2') \rangle$. It is straightforward to show that $T \cong S_5$. Then, for any $g \in T$, g acts on $V_i (i = 1, 2)$ as follows:

$$\begin{aligned} (\prod_{i=1}^5 e_i^{\lambda_i})^g &= \prod_{i=1}^5 e_{ig^{-1}}^{\lambda_i}, \text{ where } 0 \leq \lambda_i \leq 4 \text{ for each } i, \\ (\prod_{i=1}^5 e_{i'}^{\lambda_{i'}})^g &= \prod_{i=1}^5 e_{i'g^{-1}}^{\lambda_{i'}}, \text{ where } 0 \leq \lambda_{i'} \leq 4 \text{ for each } i'. \end{aligned}$$

Let $\bar{e} = \prod_{i=1}^5 e_i$, and $\bar{e}' = \prod_{i=1}^5 e_{i'}$. Let $\bar{e}_i = e_i \langle \bar{e} \rangle$ and $\bar{e}_{i'} = e_{i'} \langle \bar{e}' \rangle$ for $1 \leq i \leq 5$. Set

$$W = \langle w_1 \rangle \times \langle w_2 \rangle \times \langle w_3 \rangle \times \langle w_4 \rangle \text{ where } w_i = \bar{e}_i \bar{e}_{i'}^{-1}.$$

Then $W \cong \mathbb{Z}_p^4$. Note that T fixes each element of $\langle \bar{e} \rangle$ and $\langle \bar{e}' \rangle$. So T induces a faithful action on W . Without loss of generality, we may assume that T is a subgroup of $\text{GL}(W)$. Let $g = (11') \cdots (55')$. Obviously, g inverts each non-identity element of W .

Construction 3.11. Let $G = W:H$ where $H = \langle h, g \rangle$ with $h = (12345)(1'2'3'4'5')$. Set $X = W:N = W:(T \times \langle g \rangle) \cong \mathbb{Z}_p^4:(S_5 \times \mathbb{Z}_2)$. Let $R = \langle (1234)(1'2'3'4'), (12)(1'2') \rangle \cong S_4$, and let $y = w_1 w_5 (15)(1'5')g$. Set

$$\Gamma(p, 4, 10) = \text{Cos}(X, R, RyR).$$

Arguing similarly as Lemma 3.10, we have the following conclusion in next lemma.

Lemma 3.12. *Let $\Gamma = \Gamma(p, 4, 10)$ be a graph constructed in Construction 3.11. Then Γ is a connected tetravalent $(X, 2)$ -arc-transitive Cayley graph of Frobenius group G , and G is not normal in X . In particular, Γ is a cover of Γ_W , and $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$.*

Here are a few of graphs whose automorphism groups are almost simple.

Example 3.13. Let $\mathbb{F} = \text{GF}(p)$ be a finite field of order p . Let U and V consist of 1-subspaces and 2-subspaces of \mathbb{F}^3 , respectively.

Case 1: Let $p = 2$. Define a bipartite graph Γ with bipartite U and V such that $u \in U$ and $v \in V$ are adjacent if and only if $u + v = \mathbb{F}^3$. This is the point-line non-incidence graph of the Fano plane $\text{PG}(2, 2)$. Furthermore, $\text{Aut}\Gamma = \text{PGL}(3, 2).\mathbb{Z}_2$, and Γ is a Cayley graph of $G = D_{14}$. For example, refer to [25].

Case 2: Let $p = 3$. Define a bipartite graph Γ with bipartite U and V such that $u \in U$ and $v \in V$ are adjacent if and only if u is a subspace of v . Then Γ is the point-line incidence graph of the projective plane $\text{PG}(2, 3)$. Furthermore, $\text{Aut}\Gamma = \text{PGL}(3, 3).\mathbb{Z}_2$, and Γ is a Cayley graph of $G = D_{26}$. Refer to [14, 15], for example.

Example 3.14. Let $X = \text{PGL}(2, 7)$. By the Atlas [4], X has a maximal subgroup $H \cong D_{16}$. Pick a subgroup $K \leq H$ with $K \cong \mathbb{Z}_2^2$. Then $D_8 \cong \mathbf{N}_H(K) \leq \mathbf{N}_X(K) \cong S_4$. Choose an involution $o \in \mathbf{N}_H(K) \setminus K$ and an element $z \in \mathbf{N}_X(K)$ of order 3 such that $z^o = z^{-1}$. Then $\langle o, z \rangle \cong S_3$, and $o(oz) = 2$. Since H is a maximal subgroup of X , it follows that $\langle H, oz \rangle = X$. Let $\Gamma = \text{Cos}(X, H, HozH)$. By the choices of o and z , we conclude that $|H:H \cap H^{oz}| = 4$, namely, Γ is a connected X -arc-transitive graph of valency 4. By MAGMA [2], we have that $X = GH$ where $G = \mathbb{Z}_7:\mathbb{Z}_3$, and thus Γ is a connected X -arc-transitive Cayley graph of G of valency 4. By Li et al.[17], $\text{Aut}\Gamma = \text{PGL}(2, 7)$.

Example 3.15. Let $X = \text{PGL}(2, 7)$. Then $T = \text{soc}(X) \cong \text{PSL}(2, 7)$. Take $H \leq T$ such that $H \cong D_8$. Choose an involution o such that o is not in the center of H . It is simple to check that $\mathbf{N}_H(\langle o \rangle) \cong \mathbb{Z}_2^2$, $\mathbf{N}_T(\langle o \rangle) \cong D_8$ and $\mathbf{N}_X(\langle o \rangle) \cong D_{16}$. Let $\mathbf{N}_X(\langle o \rangle) = \mathbf{N}_T(\langle o \rangle):\langle z \rangle$ for some involution $z \in X \setminus T$. Take $y \in \mathbf{N}_T(\langle o \rangle):\langle z \rangle$ of order 4. Set $\Gamma = \text{Cos}(X, H, HxH)$, where $x = z$ or yz . By Li et al.[17], Γ is a connected tetravalent arc-transitive Cayley graph of $\mathbb{Z}_7:\mathbb{Z}_6$, and $\text{Aut}\Gamma = \text{PGL}(2, 7)$.

Example 3.16. Let $X = \text{PGL}(2, 11)$ or $\text{PSL}(2, 23)$. By the Atlas [4], X has a maximal subgroup $H \cong S_4$. Let $L \cong S_3$ be a subgroup of H . Checking the subgroups of X in the Atlas [4], we conclude that $\mathbf{N}_X(L) = \langle o \rangle \times L \cong D_{12}$, where $o \in \mathbf{N}_X(L) \setminus H$ is an involution. Set $\Gamma = \text{Cos}(X, H, HoH)$. Since H is a maximal subgroup of X , $\langle o, H \rangle = X$. It is straightforward to check that $|H:H \cap H^o| = 4$. Then Γ is a connected tetravalent X -arc-transitive graph. Moreover, X has a subgroup G which is regular on the vertices, where $G \cong \mathbb{Z}_{11}:\mathbb{Z}_5$ or $\mathbb{Z}_{23}:\mathbb{Z}_{11}$, respectively. We denote by $P_{11,5}$ and $P_{23,11}$ the graphs associated with $\text{PGL}(2, 11)$ and $\text{PSL}(2, 23)$, respectively. By Li et al.[17], $\text{Aut}P_{11,5} = \text{PGL}(2, 11)$ and $\text{Aut}P_{23,11} = \text{PSL}(2, 23)$.

Example 3.17. Let $\Gamma = (V\Gamma, E\Gamma)$ be a connected arc-transitive Cayley graph. The standard double cover $\Gamma^{(2)}$ is the graph with vertex set $V\Gamma \cup \{u'|u \in V\Gamma\}$ such that $\{u, v'\} \in E\Gamma^{(2)}$ whenever $\{u, v\} \in E\Gamma$. For each $x \in \text{Aut}\Gamma$, define $\tilde{x} : u \rightarrow u^x$, $u' \rightarrow (u^x)'$. Then $\text{Aut}\Gamma$ can be viewed as a subgroup of $\text{Aut}\Gamma^{(2)}$ in this way. Define $\epsilon : u \rightarrow u'$, $u' \rightarrow u$. Then $\epsilon \in \text{Aut}\Gamma^{(2)}$. Set $X = \langle \text{Aut}\Gamma, \epsilon \rangle$. Then $X = \text{Aut}\Gamma \times \langle \epsilon \rangle$. So $\Gamma^{(2)}$ is an X -arc-transitive Cayley graph. By Li et al.[17], $P_{11,5}^{(2)}$ is a Cayley graph of $\mathbb{Z}_{11}:\mathbb{Z}_{10}$, and $\text{Aut}P_{11,5}^{(2)} \cong \text{PGL}(2, 11) \times \mathbb{Z}_2$.

4. SOLUBLE AUTOMORPHISM GROUPS

In this section, let $G = W:H \cong \mathbb{Z}_p^d:\mathbb{Z}_n$ be a primitive Frobenius group. Let $\Gamma = \text{Cay}(G, S)$ be a connected X -edge-transitive tetravalent Cayley graph, where $G \leq X \leq \text{Aut}\Gamma$. Denote by F the Fitting subgroup of X . If X is solvable, then an important property of its Fitting subgroup is that it is self-centralized, that is, $\mathbf{C}_X(F) \leq F$. In what follows, we will determine the graph Γ for the case where X is solvable.

Lemma 4.1. *Assume that F is a r -group, where r is a prime. If Γ is a cover of Γ_F , then $F = W$.*

Proof. Note that W is minimal and normal in G . Then either $W \leq F$ or $F \cap G = 1$. If $W \leq F$, then $F = W$ as Γ is a cover of Γ_F . Thus we assume that $F \cap G = 1$.

Let $\overline{G} = G\Phi(F)/\Phi(F)$, and let $\overline{F} = F/\Phi(F)$. Since $\Phi(F) \text{ char } F$, we obtain \overline{G} can act on \overline{F} by conjugation. Clearly, $\overline{G} \cong G$. In what follows, write $\overline{G} = \overline{W}:\overline{H} \cong W:H$.

Assume first that $r \neq p$. If \overline{W} acts trivially on \overline{F} , then W induces the identity on \overline{F} . From [11, p.174, Theorem 1.4], it follows that W acts trivially on F . So $W \leq \mathbf{C}_X(F) \leq F$, against our assumption. Thus \overline{W} acts nontrivially on \overline{F} .

Let $M = \overline{F}:\overline{G}$. Let

$$1 \trianglelefteq M_1 \trianglelefteq M_2 \trianglelefteq \cdots \trianglelefteq M_{m-1} \trianglelefteq M_m = \overline{F}$$

be the normal series of \overline{F} such that each M_i/M_{i-1} is a minimal normal subgroup of M/M_{i-1} for $1 \leq i \leq m$, where $M_0 = 1$.

It is straightforward to show that \overline{G} normalizes $\mathbf{C}_{M_1}(\overline{W})$, and hence $\mathbf{C}_{M_1}(\overline{W}) \trianglelefteq M$. By the minimality of M_1 , we conclude that either $\mathbf{C}_{M_1}(\overline{W}) = 1$ or $\mathbf{C}_{M_1}(\overline{W}) = M_1$. If $\mathbf{C}_{M_1}(\overline{W}) = 1$, so that by [13, Theorem 2.7] we have $|M_1| = |\mathbf{C}_{M_1}(\overline{H})|^{|\overline{H}|}$. However, Γ is a normal cover of Γ_F , we conclude that $|M_1|$ divides $|\overline{H}|$, a contradiction occurs. Thus $\mathbf{C}_{M_1}(\overline{W}) = M_1$, that is, $\overline{W} \leq \mathbf{C}_M(M_1)$. Repeating the above argument for M_i/M_{i-1} and $\overline{G}M_{i-1}/M_{i-1}$, we obtain that $[M_i, \overline{W}] \subseteq M_{i-1}$ for $1 \leq i \leq m$. It follows that \overline{W} stabilizes the normal series of \overline{F} , and hence \overline{W} centralizes \overline{F} , refer to [11, p.178, Theorem 3.2]. It implies that W induces the identity on \overline{F} , and so W is trivial on F (see [11, p.174, Theorem 1.4]), namely, $W \leq \mathbf{C}_X(F)$, again against our assumption.

Assume now that $r = p$. Then $|F| \leq |W|$. Denote by Σ the normal quotient graph Γ_F . If $|F| = |W|$, then W fixes each vertex of Σ , and hence $W \leq F$, which is impossible. Thus $|F| < |W|$. Set $\overline{X} = X/F$. Let $F_{\overline{X}}$ be the Fitting subgroup of \overline{X} . It is known that $F_{\overline{X}}$ is a p' -group. Let $\tilde{G} = GF/F \cong G$. Write $\tilde{G} = \tilde{W}:\tilde{H}$. For that case, we conclude $F_{\overline{X}} \cap \tilde{G} = 1$. It follows that Γ is $(X, 2)$ -arc-transitive, and so Σ is $(\overline{X}, 2)$ -arc-transitive. By [26, Theorem 4.1], Σ is a cover of $\Sigma_{F_{\overline{X}}}$ or $\Sigma_{F_{\overline{X}}} = K_2$. For the former, arguing as above, we also obtain $\tilde{W} \leq F_{\overline{X}}$, which contradicts $F_{\overline{X}} \cap \tilde{G} = 1$. For the latter, we obtain $p = 2$, and $|G|$ divides $2^5 3^6$. Since G is a primitive Frobenius group, we have $G \cong \mathbb{Z}_2^2:\mathbb{Z}_3$. So $F \cong \mathbb{Z}_2$, and thus $F \leq \mathbf{Z}(X)$, again a contradiction. Therefore, $F = W$. \square

For a group T and a prime q , by T_q we mean a Sylow q -subgroup of T .

Lemma 4.2. *Use the notation defined above, rewrite $\Gamma = (V\Gamma, E\Gamma)$. Then we have:*

- (i) *If p is an odd prime, then either $G \cong D_{2p}$ or $W \trianglelefteq X$;*
- (ii) *If $p = 2$, then $F = O_2(X)$, and*
 - (a) *$W < F$, Γ_F is a cycle, and $X = (F:H).\mathcal{O}$, where $\mathcal{O} = 1$ or \mathbb{Z}_2 ;*

(b) $W = F$ and $W \trianglelefteq X$.

Proof. Suppose that $G \not\cong D_{2p}$. We first claim that $W \leq F$ and $F \cap H = 1$. Suppose, by way of contradiction, that $G \cap F = 1$. Since $X = GX_1$, we obtain $|F|$ divides $|X_1|$. From Lemma 2.2, it follows that each prime divisor of $|F|$ is either 2 or 3.

Let K be the kernel of X acting on Γ_F . Then $X/K \leq \text{Aut}\Gamma_F$. Recall that p is a prime divisor of $|W|$. Suppose that $p > 3$. Let B be an orbit of F acting on $V\Gamma$. So $|B|$ divides $|F|$. If $G \cap K \neq 1$, then $W \leq K$. Let Δ be an orbit of W acting on $V\Gamma$, which is contained in the block B . Then $|\Delta|$ divides $|B|$, which is impossible. So $G \cap K = 1$. Let $\overline{G} = GK/K$. Then $\overline{G} \cong G$ is a Frobenius group. Write $\overline{G} = \overline{W}:\overline{H} \cong G$. If Γ_K is a cycle, then $d = 1$, and so $\overline{G} \cong D_{2p}$, against our assumption. Thus Γ is a cover of Γ_K , and then $K = F$. By Lemma 2.2, $\overline{G}_B \leq \overline{H}$. In view of Lemma 4.1, F is a $\{2, 3\}$ -group. So is \overline{G}_B because $|\overline{G}_B| = |F|$. Note that Γ_F is \overline{G} -vertex-transitive. By [16, Lemma 2.1], we conclude that $\overline{G}_B^{\Gamma(B)}$ is a cyclic group of order $2^i 3^j$ for $i, j \geq 1$. However, $\overline{G}_B^{\Gamma(B)}$ is isomorphic to a subgroup of S_4 , which is a contradiction. Thus $p = 2$ or 3.

If $F = O_2(X)$, by Lemma 4.1, Γ_F is a cycle. So is Γ_K . By the assumption, we conclude that $p = 2$, and $W \leq K$. It follows that $K = W.K_1$, where K_1 is a 2-group. Since $K \trianglelefteq X$, we have $K \leq F$, which contradicts the fact that $F \cap G = 1$.

If $F = O_3(X)$, then Γ is $(X, 2)$ -arc-transitive. From Lemma 4.1, it follows that $\Gamma_F = K_2$. It implies that $G \cong D_6$, against our assumption.

If $F = O_2(X) \times O_3(X)$, then Γ is $(X, 2)$ -arc-transitive. By [26, Theorem 4.1], Γ is a cover of Γ_F , $\Gamma_F = K_2$ or F is transitive on $V\Gamma$. Assume first that Γ is a cover of Γ_F . Let $Y = \text{Aut}\Gamma_F$ and $\overline{X} = X/F$. Then \overline{X} is a subgroup of Y . Recall that $F \cap G = 1$, we have $|\overline{G}_B| = |F|$, and hence \overline{G}_B is a Frobenius group of Y_B . Since Γ_F is \overline{G} -vertex-transitive, we conclude that $\overline{G}_B^{\Gamma_F(B)} = A_4, S_3$ or S_4 , refer to [16, Lemma 2.1]. Assume $\overline{G}_B^{\Gamma_F(B)} = A_4$ or S_4 . Then Γ_F is $(\overline{G}, 2)$ -arc-transitive. From Lemma 2.2, it follows that $\overline{G}_B = A_4$, which implies that $|F| = 12$. For this case, it is easy to show that $W \leq \mathbf{C}_X(F) \leq F$, a contradiction occurs. Assume $\overline{G}_B^{\Gamma_F(B)} = S_3$. Let $\overline{G}_B^{[1]}$ be the kernel of \overline{G}_B acting on $\Gamma_F(B)$. Recall that \overline{G}_B is a Frobenius group, we conclude $\overline{G}_B^{[1]}$ is a 3-group, and hence $O_2(X) \cong \mathbb{Z}_2$. It further implies that W is a 3-group. Let \overline{F} be the Fitting subgroup of \overline{X} . Then $\overline{F} \cap \overline{G} = 1$. It follows that \overline{F} is a 2-group. Since $|F.\overline{F}|$ divides $2^4 3^6$, we have $|\overline{F}| \leq 8$, and thus $\overline{W} \leq \mathbf{C}_{\overline{X}}(\overline{F})$, again a contradiction.

Assume now that $\Gamma_F = K_2$. Then $|V\Gamma|$ divides $2^5 3^6$. Since Γ is a cover of Γ_{F_2} and Γ_{F_3} , we conclude that $|F_2| = \frac{|G_2|}{2}$ or $|G_2|$, and $|F_3| = |G_3|$. When $p = 3$, we have $|F_3| = |W|$. Note that W is minimal and normal in G . Then W fixes each vertex of Γ_{F_3} , and thus $W \leq F$, which contradicts $G \cap F = 1$. For $p = 2$, and $|F_2| = |G_2|$, we also obtain the same contradiction. When $p = 2$ and $|F_2| = \frac{|G_2|}{2}$. Since G is a $\{2, 3\}$ -group, we conclude that $G \cong \mathbb{Z}_2^2:\mathbb{Z}_3$, and hence $|F| = 6$. For that case, we easily obtain that $W \leq F$, again a contradiction. Similarly, we also exclude the case where F is transitive on $V\Gamma$.

Summarizing the above discussion, we obtain $W \leq F$. Since G is a Frobenius group, we have $F \cap H = 1$, as claimed. Next we process our analysis by several cases.

Case 1: If $p > 3$, then $W \trianglelefteq X$.

By the previous discussion, we have $W \leq F_p$. By Lemma 2.2, we conclude that $W = F_p$, and hence $W \trianglelefteq X$.

Case 2: If $p = 3$, then $W \trianglelefteq X$.

If $W < F_3$, then Γ is $(X, 2)$ -arc transitive. For this case, $\Gamma_{F_3} = K_2$, and so $G \cong D_6$, contrary to our assumption. Thus $W = F_3$, and then $W \trianglelefteq X$.

Case 3: If $p = 2$, then either Γ_F is a cycle, or $W \trianglelefteq X$.

Assume that $W < F_2$. Since $F \cap H = 1$, we know that $|FH| = |F||H|$. Note that $|FH|$ divides $|X|$, it follows that $|F|$ divides $|W||X_1|$, and hence $|F_{2'}|$ divides $|X_1|$. So $F_{2'}$ is a 3-group. If $F_{2'} \neq 1$, then Γ is $(X, 2)$ -transitive, and thus it follows from [26, Theorem 4.1] that Γ is a cover of Γ_{F_2} , a contradiction. So $F_{2'} = 1$. By Lemma 4.1, Γ_F is a cycle because $|H|$ is an odd number. Recall that B is a vertex of Γ_F . Since Γ is a Cayley graph of G , we obtain W is regular on B . So $K = F = WK_1$. Consequently, $X = (F:H).\mathcal{O}$ where $\mathcal{O} \cong 1$ or \mathbb{Z}_2 . This completes the proof of Lemma 4.2. \square

For the group $G \cong D_{2p}$ where p is an odd prime. Applying [17, Theorem 1.1], we have the following conclusions.

Lemma 4.3. *Let $G \cong D_{2p}$, and let Γ be a connected edge-transitive tetravalent Cayley graph of G . Then we have*

- (i) Γ is arc-regular, and $\text{Aut}\Gamma \cong D_{2p}:\mathbb{Z}_4$;
- (ii) $\Gamma \cong \mathbf{C}_{p[2]}$, and $\text{Aut}\Gamma \cong \mathbb{Z}_2^p:D_{2p}$.

In the remainder of this section assume that $G \not\cong D_{2p}$ with p an odd prime, unless specified otherwise.

Recall that the *socle* of a finite group R (denoted by $\text{soc}(R)$) is the product of all minimal normal subgroups of R . Evidently, $\text{soc}(R)$ is a characteristic subgroup of R .

We next treat the case where $W \trianglelefteq X$, and the normal quotient Γ_W is a cycle.

Lemma 4.4. *Let K be the kernel of X acting on Γ_W . Then the following statements hold:*

- (i) $X = ((WK_1):H).\mathcal{O}$, and $W \cong \mathbb{Z}_2^d$, where $\mathcal{O} \cong 1$ or \mathbb{Z}_2 ;
- (ii) Assume p is an odd prime. Then we have
 - (1) G is normal in X , or
 - (2) G is not normal in X , and
 - (a) $X = W:((K_1:H).\mathcal{O})$, and H does not centralise K_1 where $K_1 \cong \mathbb{Z}_2^l$ with $2 \leq l \leq d$, and $\mathcal{O} \cong 1$ or \mathbb{Z}_2 ;
 - (b) there exist $x_1, x_2, \dots, x_d \in W$ and $\tau_1, \tau_2, \dots, \tau_d \in K_1$ such that $W = \langle x_1, \dots, x_d \rangle$, $\langle x_i, \tau_i \rangle \cong D_{2p}$, and $K_1 = \langle \tau_i \rangle \times \mathbf{C}_{K_1}(x_i)$ for $1 \leq i \leq d$;
 - (c) $\text{soc}(X) = W \times L$, where $L \cong 1$ or \mathbb{Z}_2 ;
 - (d) H is imprimitive on W .

Proof. Let B be a vertex of Γ_W . Since Γ is a Cayley graph of G , we obtain W is regular on B . Thus $K = WK_1$, and $K \cap H = 1$, where K_1 is a 2-group. For that case, Γ_W is a connected Cayley graph. Recall that H is of order n , Γ_W is a cycle of size n , say. It follows that $X/K \cong \mathbb{Z}_n$ or D_{2n} . Further, Γ is X -arc-transitive if and only if $X/K \cong D_{2n}$.

Assume first that $p = 2$. Since $G \leq X$ and $(|K|, |H|) = 1$, we conclude that $K:H \leq X$. Noting that X/K is isomorphic to a subgroup of D_{2n} , it follows that $X = (K:H).\mathcal{O}$ with $\mathcal{O} \cong 1$ or \mathbb{Z}_2 , so we have part (i).

Assume now that p is an odd prime. Furthermore, we assume that G is not normal in X . If $K_1 = 1$, then $K = W$, and hence $G \triangleleft X$, which contradicts the assumption. Thus $K_1 \neq 1$.

Let $U = \mathbf{N}_X(K_1)$. Since $K_1 \not\trianglelefteq X$, it follows that $U \neq X$. Noticing that $(|W|, |K_1|) = 1$, we obtain that $\mathbf{N}_{X/W}(K/W) = \mathbf{N}_{X/W}(WK_1/W) = \mathbf{N}_X(K_1)W/W = UW/W$. As $K/W \trianglelefteq X/W$, implying that $X = WU$. Since $W \triangleleft X$, we have that $W \cap U \triangleleft U$. Furthermore, $W \cap U \triangleleft W$ since W is abelian. Then $W \cap U \triangleleft \langle U, W \rangle = UW = X$. If $W \leq U$, then $K = WK_1 = W \times K_1$, and hence $K_1 \triangleleft X$, which is impossible. Thus $W \cap U < W$. Furthermore, note that W is a minimal normal subgroup of X , we obtain that $W \cap U = 1$, and so $K \cap U = WK_1 \cap U = (W \cap U)K_1 = K_1$. Now $X/K = UW/K = UK/K \cong U/(K \cap U) = U/K_1$, and hence $U = (K_1 \cdot \hat{H}) \cdot \mathcal{O}$, where $\hat{H} \cong \mathbb{Z}_n$ and $\mathcal{O} \cong 1$ or \mathbb{Z}_2 . Noting that G belongs to X and $G = W:H$, there exists some $H^z \leq U$ such that $G = W:H^z$ is regular on $V\Gamma$, where $z \in W$. Without loss of generality, we may assume that $U = (K_1:H) \cdot \mathcal{O}$. Then $X_1 = K_1 \cdot \mathcal{O}$. Furthermore, since G is not normal in X , we conclude that H does not centralise K_1 .

Set $Y = W:(K_1:H)$. Then Y has index at most 2 in X , and Γ is Y -edge-transitive. It is obvious that Γ is not Y -arc-transitive. Hence $\Gamma = \text{Cos}(Y, K_1, K_1\{y, y^{-1}\}K_1)$, where $y \in Y$ is such that $\langle K_1, y \rangle = Y$ and $K_1 \cap K_1^y$ has index 2 in K_1 . We may choose $y \in W:H = G$ such that $H = \langle h \rangle$ and $y = hx$ where $x \in W$. Then $K_1 \cap K_1^y = K_1 \cap K_1^x$ has index 2 in K_1 .

We claim that $K_1 \cap K_1^x = \mathbf{C}_{K_1}(x)$. For any $\sigma \in K_1 \cap K_1^x$, we have that $\sigma^{x^{-1}} \in K_1$, and hence $\sigma^{-1}\sigma^{x^{-1}} \in K_1$. Since $x \in W$ and $W \triangleleft WK_1$, we obtain that $\sigma^{-1}\sigma^{x^{-1}} = (\sigma^{-1}x\sigma)x^{-1} \in W$. So $\sigma^{-1}\sigma^{x^{-1}} \in W \cap K_1 = 1$, and then $\sigma^{x^{-1}} = \sigma$. Thus σ centralises x . It follows that $K_1 \cap K_1^x \leq \mathbf{C}_{K_1}(x)$. Clearly, $\mathbf{C}_{K_1}(x) \leq K_1 \cap K_1^x$. So $\mathbf{C}_{K_1}(x) = K_1 \cap K_1^x$ as required.

Recall that W is a minimal normal subgroup of X and $X = WU$, we obtain that $W = \langle x \rangle \times \langle x^{\sigma_2} \rangle \times \cdots \times \langle x^{\sigma_d} \rangle$ where $\sigma_i \in U$. Then $\mathbf{C}_{K_1}(x^{\sigma_i}) = (\mathbf{C}_{K_1}(x))^{\sigma_i} < K_1^{\sigma_i} = K_1$. The intersection $\cap_{i=1}^d \mathbf{C}_{K_1}(x^{\sigma_i}) \leq \mathbf{C}_K(W) = W$, and hence $\cap_{i=1}^d \mathbf{C}_{K_1}(x^{\sigma_i}) = 1$. Since each $\mathbf{C}_{K_1}(x^{\sigma_i})$ is a maximal subgroup of K_1 , the Frattini subgroup $\Phi(K_1) \leq \cap_{i=1}^d \mathbf{C}_{K_1}(x^{\sigma_i}) = 1$. Hence K_1 is an elementary abelian 2-group, that is, $K_1 \cong \mathbb{Z}_2^l$ for some $l \geq 1$. Recall that $\cap_{i=1}^d \mathbf{C}_{K_1}(x^{\sigma_i}) = 1$, it follows that $l \leq d$. Assume that $l = 1$. Then $K_1 \cong \mathbb{Z}_2$ and so $K_1 \leq \mathbf{C}_X(H)$. Thus $G \triangleleft X$, which contradicts the fact that G is not normal in X . Hence $l > 1$, as in part (a).

Since $\mathbf{C}_{K_1}(x)$ has index 2 in K_1 , there exists some τ_1 belonging to K_1 such that $K_1 = \langle \tau_1 \rangle \times \mathbf{C}_{K_1}(x)$. Set $x_1 = x^{-1}x^{\tau_1}$. Then $x_1 \neq 1$, $x_1^{\tau_1} = x_1^{-1}$ and $\mathbf{C}_{K_1}(x) = \mathbf{C}_{K_1}(x_1)$, and hence $K_1 = \langle \tau_1 \rangle \times \mathbf{C}_{K_1}(x_1)$. Noticing that W is a minimal normal subgroup of $X = WU$, there exist $\mu_1 = 1, \mu_2, \dots, \mu_d \in U$ such that $W = \langle x^{\mu_1} \rangle \times \langle x^{\mu_2} \rangle \times \cdots \times \langle x^{\mu_d} \rangle$. Let $x_i = x_1^{\mu_i}$ and $\tau_i = \tau_1^{\mu_i}$, where $i = 1, \dots, d$. Then $\mathbb{Z}_2^{l-1} \cong (\mathbf{C}_{K_1}(x_1))^{\mu_i} = \mathbf{C}_{K_1^{\mu_i}}(x_1^{\mu_i}) = \mathbf{C}_{K_1}(x_i)$, and $K_1 = K_1^{\mu_i} = \langle \tau_i \rangle \times \mathbf{C}_{K_1}(x_i)$. Furthermore, $x_i^{\tau_i} = x_1^{\tau_1 \mu_i} = (x_1^{-1})^{\mu_i} = x_i^{-1}$, and so $\langle x_i, \tau_i \rangle \cong D_{2p}$, as in part (b).

Recall that $W \cong \mathbb{Z}_p^d$ for an odd prime p . Since G is not normal in X , we conclude that $d > 1$. Assume that X has a minimal normal subgroup $L \neq W$. Then $W \cap L = 1$, and $LK/K \triangleleft X/K \leq D_{2n}$. It follows that either $L \leq K$, or $L \cap K = 1$. If $L \leq K$, then L is a 2-group. Since K_1 is a Sylow 2-subgroup of K , there exists some $w \in W$ such that $L^w \leq K_1$. It follows that $L \trianglelefteq K_1$, and then $L = 1$, which is impossible. Hence $L \cap K = 1$, and so $L \leq K_1H$, and $L \cong \mathbb{Z}_2$. Thus $\text{soc}(X) = W \times L$, as in part (c).

By the above paragraph, we obtain that $\mathbf{C}_X(W) = W \times L$. Let $\overline{X} = X/L$, and $\overline{G} = GL/L \cong G$. Let $\overline{K}_1 = K_1L/L \cong K_1$. Write $\overline{G} = \overline{W}:\overline{H}$. Since H normalizes K_1 , we conclude that \overline{H} normalizes \overline{K}_1 . Note that $\overline{K}_1\overline{H}$ acts irreducibly and faithfully on

\overline{W} . By Clifford's Theorem, \overline{W} can be decomposed as $\overline{W} = e(U_1 \oplus U_2 \oplus \cdots \oplus U_t)$ such that \overline{K}_1 normalises each U_i , and all U_i are pairwise non-equivalent and irreducible with respect to the action of \overline{K}_1 . Recall that K_1 is of order at least 4. It implies that $t \geq 2$. Let $V_i = eU_i$ for each i . Rewrite $\overline{W} = V_1 \oplus V_2 \oplus \cdots \oplus V_t$. Now \overline{H} normalises \overline{K}_1 , we conclude that \overline{H} preserves such decomposition. Since the maximal subgroup preserving such decomposition in $\text{GL}(\overline{W})$ is $\text{GL}(V_1) \wr S_t$, implying that \overline{H} belongs to $\text{GL}(V_1) \wr S_t$, forcing \overline{H} is imprimitive on \overline{W} . By [6, Proposition 2.8], we are done, as in part (d). \square

We now determine the graph Γ for the case where $W \trianglelefteq X$, and Γ is a normal cover of Γ_W .

Lemma 4.5. *Assume that Γ is a normal cover of Γ_W . Then we have*

- (i) G is normal in X , or
- (ii) G is not normal in X , and
 - (a) $\Gamma_W \cong \mathbf{C}_{\frac{n}{2}[2]}$, and $n \equiv 0 \pmod{4}$;
 - (b) $X = W:((NH).\mathcal{O})$, $X_1 \leq N.\mathcal{O}$, $N \cap H \cong \mathbb{Z}_2$, and H normalizes N , but H does not centralise N , where $N \cong \mathbb{Z}_2^l$ with $2 \leq l \leq \frac{n}{2}$, and $\mathcal{O} \cong 1$ or \mathbb{Z}_2 ;
 - (c) W is unique and minimal in X , and H is imprimitive on W ;
 - (d) $X/(WN) \cong \mathbb{Z}_{\frac{n}{2}}$ or D_n , and Γ is X -arc-transitive if and only if $X/(WN) \cong D_n$.

Proof. Let $\overline{H} = G/W$ with $\overline{H} = \langle \overline{h} \rangle$. Since Γ is a Cayley graph of G , Γ_W is a Cayley graph of \overline{H} . By [1, Theorem 1.2], either Γ_W is a normal Cayley graph or $\Gamma_W = \mathbf{C}_{\frac{n}{2}[2]}$, and 4 divides n . It follows that either G is normal in X or $\Gamma_W \cong \mathbf{C}_{\frac{n}{2}[2]}$. Suppose that G is not normal in X . Then $\Gamma_W \cong \mathbf{C}_{\frac{n}{2}[2]}$, as in part (a).

Clearly, $\text{Aut} \Gamma_W \cong \mathbb{Z}_2^{\frac{n}{2}}:D_n$. Let $\overline{K} \trianglelefteq \text{Aut} \Gamma_W$ such that $\overline{K} \cong \mathbb{Z}_2^{\frac{n}{2}}$. Then we may write $\text{Aut} \Gamma_W = \overline{K} \overline{H} \mathcal{O}$, where $\mathcal{O} \cong \mathbb{Z}_2$. Let B be a vertex of Γ_W , and $1 \in B$. Choose $\overline{M} \leq \overline{K}$ such that $M \cong \mathbb{Z}_2^{\frac{n}{2}-1}$ and $(\text{Aut} \Gamma_W)_B = \overline{M} \mathcal{O}$.

Let $\overline{X} = X/W$. Since $\overline{X} \overline{K}/\overline{K} \cong \overline{H} \mathcal{O}/(\overline{H} \mathcal{O} \cap \overline{K})$ where $\mathcal{O} = 1$ or \mathcal{O} , we conclude that $\overline{X} = (\overline{X} \cap \overline{K}) \overline{H} \mathcal{O}$, and Γ is X -arc-transitive if and only if $\mathcal{O} = \mathcal{O}$. Let $\hat{K} = \overline{X} \cap \overline{K}$. Then $\hat{K} \trianglelefteq \overline{X}$, and $\hat{K} \cap \overline{H} \cong \mathbb{Z}_2$. Thus $X = W.((\hat{K} \overline{H}).\mathcal{O})$. Let K be the preimage of \hat{K} , under $X \rightarrow X/W$. Note that G is a Frobenius group, we conclude that the order of W is odd. By Hall's Theorem, $K = W:N$, where $N \cong \hat{K}$. It further implies that $N \cong \mathbb{Z}_2^l$, where $l \leq \frac{n}{2}$.

Now $(|N|, |W|) = 1$, we get $X/W = \mathbf{N}_{X/W}(NW/W) = \mathbf{N}_X(N)W/W$, and so $X = W\mathbf{N}_X(N)$. Since $H \leq X$, it follows that H^w belongs to $\mathbf{N}_X(N)$ for some $w \in W$. Without loss of generality, we may assume that H belongs to $\mathbf{N}_X(N)$. Thus $X = W:((NH).\mathcal{O})$. By comparing the order, we conclude that $N \cap H \cong \mathbb{Z}_2$. If $l = 1$, then $NH = H$, and so $G \trianglelefteq X$, which contradicts the assumption that G is not normal in X . Then $l \geq 2$. Thus $2 \leq l \leq \frac{n}{2}$.

Set $Y = W:(NH)$. Clearly, G is a subgroup of Y . Then $Y = GY_1$. Note that $|Y| = \frac{|W||H||N|}{|H \cap N|}$, and $|Y| = |G||Y_1|$, we obtain $|Y_1| = \frac{|N|}{|N \cap H|} = \frac{|N|}{2}$. Let $\overline{Y} = Y/W$. Since $Y_1W/W = \overline{Y}_B$, we have $Y_1W/W \leq \overline{Y} \cap \overline{M} \leq \hat{K}$, and hence $Y_1W/W \leq NW/W$. Consequently, $Y_1 \leq N^{\overline{w}}$ for some $\overline{w} \in W$. For simplicity, we may assume that $Y_1 \leq N$. For that case, Y_1 has index 2 in N , and hence $X_1 \leq N.\mathcal{O}$, as in part (b).

Let $C = \mathbf{C}_{NH}(W)$. Assume that $C \neq 1$. Clearly, C is normal in Y . Without loss of generality, C is minimal in Y . Since H acts fixed-point-freely on W , we have $C \cap H = 1$. Let \overline{C} be the image of C under $X \rightarrow X/W$. Then \overline{C} is normal and minimal in \overline{Y} , and hence \overline{C} is a subgroup of \hat{K} . It implies that $C \cong \mathbb{Z}_2^\ell$ for some ℓ .

Let $\overline{K} = \prod_{i=1}^{\frac{n}{2}} \langle \sigma_i \rangle$. Note that \overline{H} acts on \overline{K} by permuting transitively on all σ_i . Relabeling if necessary, we may assume $\overline{h} = \sigma\pi$, where $\sigma \in \overline{K}$, and $\pi = (12 \cdots \frac{n}{2})^{-1}$. Let $\overline{K}_B = \prod_{i \neq 1} \langle \sigma_i \rangle$. Choose $\overline{B}, \tilde{B} \in \Gamma(B)$ such that $\overline{K}_{\overline{B}} = \prod_{i \neq 2} \langle \sigma_i \rangle$ and $\overline{K}_{\tilde{B}} = \prod_{i \neq \frac{n}{2}} \langle \sigma_i \rangle$. Pick some $x \in C$ such that $x = \sigma_{i_1} \cdots \sigma_{i_k}$, where $i_1 = 2$, and $2 \leq i_j \leq i_{j+1} \leq \frac{n}{2}$. Then $\langle x, x^{\overline{h}^{\frac{n}{2}-i_k}} \rangle \leq \overline{K}_B \cap C$. It follows that Γ_W is $\overline{C}:\overline{H}$ -edge-transitive Cayley graph of valency 4.

Let $Z = (W \times C):H$. By the above paragraph, Γ is Z -edge-transitive. However, Γ is not Z -arc-transitive. By Lemma 2.1, $\Gamma = \text{Cos}(Z, Z_1, Z_1\{g, g^{-1}\}Z_1)$. It is obvious that $Z_1 \cong \mathbb{Z}_2^\ell$. Now we may assume $Z_1 = \langle \tau_1, \tau_2, \dots, \tau_\ell \rangle$. Write $h = \langle (h_1, h_2) \rangle$. If $\tau_i = (1, c)$ for some i , since C is minimal in Z , we conclude that $Z_1 = C$, which is impossible. Thus each τ_i has the form $(1, c)h^{\frac{n}{2}}$ or $(u, c)h^{\frac{n}{2}}$, where $u \in W \setminus \{1\}$, and $c \in C$. Note that n is divisible by 4. Then we may write g as $(v, c')h$, where $v \in W$, and $c' \in C$. Assume $\tau_{i_0} = (u_0, c)h^{\frac{n}{2}}$ for some i_0 , where $u_0 \neq 1$. Then all τ_i have the form $(u_0, c')h^{\frac{n}{2}}$, where $c' \in C$. If $\tau_i^g = \tau_j$ for some two i, j , it is easy to show that $\langle Z_1, g \rangle$ is a subgroup of $C:\langle g \rangle$, which is a contradiction. Thus all τ_i^g do not belong to Z_1 , which leads to the valency of Γ is greater than 4, again a contradiction. Similarly, we also exclude the other case. Thus $C = 1$, namely, NH acts faithfully and irreducibly on W . That is to say, W is the unique minimal normal subgroup of X . Arguing similarly as Lemma 4.4, we obtain that H is imprimitive on W , as in part (c).

Clearly, Γ_{WN} is a cycle. Since $X/(WN)$ is transitive on $V\Gamma_{WN}$, we conclude that $X/(WN) \cong \mathbb{Z}_{\frac{n}{2}}$ or D_n . For that case, Γ is arc-transitive if and only if $X/(WN) \cong D_n$, as in part (d). \square

With the above preparation, we are ready to embark on the proof of Theorem 1.1.

Proof of Theorem 1.1: If $G \triangleleft X$, then by Lemma 2.4, we have $X_1 \leq D_8$, as in Theorem 1.1 (1). In what follows, we assume that G is not normal in X .

Assume first that p is an odd prime. By Lemmas 4.3-4.5, if W is not normal in X , we obtain that $\Gamma \cong \mathbf{C}_{p[2]}$, and $\text{Aut}\Gamma \cong \mathbb{Z}_2^p:D_{2p}$, as in Theorem 1.1 (2). If W is normal in X , and Γ_W is a cycle, by Lemma 4.4, part (3) of Theorem 1.1 occurs. If W is normal in X , and Γ is a cover of Γ_W , from Lemma 4.5, it follows that part (4) of Theorem 1.1 holds.

Assume now that $p = 2$. By Lemmas 4.2 and 4.4, Theorem 1.1 (5) occurs. \square

5. INSOLUBLE AUTOMORPHISM GROUPS

Let $G = W:H \cong \mathbb{Z}_p^d:\mathbb{Z}_n$ be a primitive Frobenius group. Assume that $\Gamma = (V\Gamma, E\Gamma)$ is a connected X -edge-transitive tetravalent Cayley graph of G , where $G \leq X \leq \text{Aut}\Gamma$. In this section, we study the case where the automorphism group X is insoluble.

For a finite group R , the socle of R , denoted by $\text{soc}(R)$, is the subgroup generated by all minimal normal subgroups of R . The group R is said to be almost simple if its socle $\text{soc}(R)$ is a non-abelian simple group.

TABLE 2: Almost simple automorphism groups.

X	G	X_1
$\text{PSL}(3, 3): \mathbb{Z}_2$	D_{26}	$\mathbb{Z}_3^2: \text{GL}(2, 3)$
$\text{PGL}(2, 7)$	D_{14}	S_4
$\text{PGL}(2, 7)$	$\mathbb{Z}_7: \mathbb{Z}_3$	D_{16}
$\text{PGL}(2, 7)$	$\mathbb{Z}_7: \mathbb{Z}_6$	D_8
$\text{PSL}(2, 23)$	$\mathbb{Z}_{23}: \mathbb{Z}_{11}$	S_4
$\text{PSL}(2, 11)$	$\mathbb{Z}_{11}: \mathbb{Z}_5$	A_4
$\text{PGL}(2, 11)$	$\mathbb{Z}_{11}: \mathbb{Z}_5$	S_4
$\text{PGL}(2, 11)$	$\mathbb{Z}_{11}: \mathbb{Z}_{10}$	A_4

We now determine the structure of insoluble group X . Denote by $R(X)$ the maximal solvable normal subgroup of X . We first treat the case where $R(X) = 1$.

Lemma 5.1. *Let N be minimal and normal in X . If $R(X) = 1$, then $\mathbf{C}_X(N) = 1$.*

Proof. Note that N is minimal in X . Since $R(X) = 1$, we have $N \cong T^k$, where T is a nonabelian simple group, and k is an integer. Clearly, $\mathbf{Z}(N) = 1$. Let $C := \mathbf{C}_X(N)$. Since $N \trianglelefteq X$, we have $C \trianglelefteq X$. Suppose that $C \neq 1$. By our assumption, C is insoluble. Notice that $N \cap G \trianglelefteq G$, we conclude that $N \cap G = 1$ or $W \leq N \cap G$. For the former, $|N|$ divides $|X_1|$, and so N is soluble, contrary to our assumption. Thus $W \leq N \cap G$. Similarly, $W \leq C \cap G$. It follows that $W \leq N \cap C$, a contradiction. Thus $C = 1$. \square

Lemma 5.2. *If $R(X) = 1$, then X is almost simple.*

Proof. By Frattini argument, we have that $X = GX_u$, where $u \in V\Gamma$. By Lemma 2.2, either X_u is a 2-group or $|X_u|$ divides $2^4 3^6$. Let N be a minimal normal subgroup of X . By our assumption, N is unsolvable. So $N = T_1 \times T_2 \times \cdots \times T_k$, where $T_i \cong T$ is a nonabelian simple group for any i . By [12], we obtain that T is one of the following:

$$\text{PSL}(2, q)(q > 3), \text{PSL}(3, q)(q < 9), \text{PSL}(4, 2), \text{PSp}(4, 3), \text{PSU}(3, 8), \text{ or } M_{11}.$$

In what follows, suppose that $k \geq 2$. Since W is minimal and normal in G and $N \cap G \neq 1$, we conclude that $W \leq N$. Let $r > 3$ be a prime divisor of $|T|$. Since r divides $|X|$ and $(|W|, |H|) = 1$, we conclude that r divides either $|W|$ or $|H|$. Suppose first that r divides $|W|$. Then $T_i \cap W \neq 1$ for each i . Let $W_i = T_i \cap W$ for $1 \leq i \leq k$. Assume that $N \cap H = 1$. Then $\frac{|N|}{|W|}$ divides $|X_u|$. So does $\prod_{i=1}^k \frac{|T_i|}{|W_i|}$. An inspection of the above simple groups shows $T = A_5$ and $k = 2$. By Lemma 5.1, $X \lesssim S_5 \wr \mathbb{Z}_2$. For this case, $\frac{|N|}{|W|} = 144$. By Lemma 2.2, the only possibility is that $X \cong A_5 \times A_5$. Clearly, this is a contradiction.

Thus $N \cap H \neq 1$. Let $\overline{H} = N \cap H$. Since G is a Frobenius group, it follows that \overline{H} is a diagonal subgroup of N . Write $\overline{H} = \langle \sigma_1 \sigma_2 \cdots \sigma_k \rangle$ where $\langle \sigma_i \rangle \cong \overline{H}$, and $\sigma_i \in T_i$ for each i . Let $H_i = \langle \sigma_i \rangle$ where $1 \leq i \leq k$. Then $G_i = W_i: H_i$ is a Frobenius group. For this case, we obtain that $\frac{|N|}{|W||\overline{H}|}$ divides $|X_u|$.

Let $T = \text{PSL}(2, q)$ where $q > 3$. By [28, Theorem 6.25], we conclude that $G_i \leq [q]:[\frac{q-1}{d}]$, $G_i \leq D_{\frac{2(q+1)}{d}}$, $G_i \leq A_5$, or $G_i \leq \text{PGL}(2, r)$, where $d = (2, q-1)$, and $r | q$. Suppose $G_i \leq [q]:[\frac{q-1}{d}]$. Then $W:\overline{H} \leq [q]^k: [\frac{q-1}{d}]$, namely, $N \cap G \leq [q]^k: [\frac{q-1}{d}]$. Since $|N:N \cap G|$ divides $|X_u|$, we conclude that $\frac{d|N|}{q^k(q-1)}$ divides $|X_u|$. If q is even, then $(q+1)^k(q-$

$1)^{k-1} \mid 2^4 3^6$, which is a contradiction because $q+1$ and $q-1$ are two distinct odd numbers. If q is odd, then $(q+1)^k (\frac{q-1}{2})^{k-1} \mid 2^4 3^6$. By easy calculations, $q = 5$ and $k = 2$. For this case, the only possibility is that $G \cong \mathbb{Z}_5^2 : \mathbb{Z}_8$ and $X \cong \mathbb{S}_5 \wr \mathbb{Z}_2$. By Lemma 2.2, we have that $X_u \cong \mathbb{S}_3 \times \mathbb{S}_4$. Without loss of generality, we may assume that $X = (\mathbb{S}_5 \times \mathbb{S}_5) : \langle \sigma \rangle$, where σ permutes the first and second coordinates. Note that $G \cap X_u = 1$. By MAGMA [2], there is an element G (up to conjugate) in X , and there are two elements X_u (up to conjugate) in X such that their intersections equal to 1. Choose $w, h \in \mathbb{S}_5$ such that $o(w) = 5$, $o(h) = 4$ and $w^h = w^2$. For this case, write $G = W : H$ with $W = \langle (w, 1), (1, w) \rangle$ and $H = (h, 1)\sigma$. Meanwhile, we choose $X_u = \langle ((123), 1), ((12), 1), (1, (1234)), (1, (12)) \rangle$ or $\langle ((123), 1), ((12)(45), 1), (1, (1234)), (1, (12)) \rangle$. It is simple to show that, for the above two choices, X_u belongs to different conjugate classes of X , and $X_u \cap G = 1$. Choose $v \in \Gamma(u)$. By Lemma 2.1, write $\Gamma = \text{Cos}(X, X_u, X_u o X_u)$, where $o \in \mathbf{N}_X(X_{uv}) \setminus X_u$ and $o^2 \in X_{uv}$. Since Γ is X -arc-transitive graph, we conclude $|X_u : X_{uv}| = 4$, and hence $|X_{uv}| = 36$. In such two cases, again by MAGMA [2], there is no $o \in \mathbf{N}_X(X_{uv})$ such that $\langle X_u, o \rangle = X$, namely, Γ is not connected. Suppose that $G_i \leq \text{D}_{\frac{2(q+1)}{d}}$ or $G_i \leq \text{A}_5$ where $1 \leq i \leq k$. Arguing similarly as above, we conclude that $q = 4$, $k = 2$, and $G_i \cong \text{D}_{10}$ for each i . For this case, the only possibility is that $G \cong \mathbb{Z}_5^2 : \mathbb{Z}_8$ and $X \cong \mathbb{S}_5 \wr \mathbb{Z}_2$, which is impossible by the above discussion. Thus $G_i \leq \text{PGL}(2, r)$. Write $q = p^n$ where p is a prime, and n is a natural number. Then $r = p^m$, where $m \mid n$. Let $n = ms$ with $s \geq 2$. Note that $|N : N \cap G|$ divides $|X_u|$. So does $|T : \text{PGL}(2, r)|^k$. It follows that $\frac{1}{2}p^{m(s-1)}(\sum_{i=1}^s p^{m(s-i)})(\sum_{j=1}^s p^{2m(s-j)})$ divides $2^4 3^6$. By easy calculations, there are no p, n and m satisfying the above equation. Thus this case is excluded.

Let $T = \text{PSL}(3, q)$ with $q < 9$. Assume that $q = 2$. By Atlas [4], we have $G_i \cong \mathbb{Z}_7 : \mathbb{Z}_3$ where $1 \leq i \leq k$. Clearly, $\frac{|N|}{|W||H|}$ does not divide $|X_u|$. Similarly, we can exclude the other cases. Let $T = \text{PSL}(4, 2)$. Again by Atlas [4], we conclude that $35 \nmid |G_i|$ where $1 \leq i \leq k$. It implies that 5 or 7 divides $|X_u|$, which is impossible. Arguing similarly as above, we can conclude T can not equal to other simple groups.

Suppose now that r divides $|H|$. If $r \nmid |H_i|$, then r divides $|X_u|$, which is impossible. Thus r divides $|H_i|$ for each i . Recall that $\frac{|N|}{|W||H|}$ divides $|X_u|$, we conclude that r divides $|X_u|$, again a contradiction. Therefore, X is almost simple. \square

Lemma 5.3. *Let X be an almost simple group with $\text{soc}(X) = \text{PSL}(2, 7)$. Assume that Γ is not $(X, 2)$ -arc-transitive. Then either $X = \text{PGL}(2, 7)$, $X_1 = \text{D}_8$ and $G = \mathbb{Z}_7 : \mathbb{Z}_6$, or $X = \text{PGL}(2, 7)$, $X_1 = \text{D}_{16}$ and $G = \mathbb{Z}_7 : \mathbb{Z}_3$.*

Proof. By Frattini argument, we have that $X = GX_u$, where $u \in V\Gamma$. Since Γ is not $(X, 2)$ -arc-transitive, it follows that X_u is a 2-group. Note that G is a Frobenius group. Checking the subgroups of $\text{PGL}(2, 7)$ in the Atlas [4], we obtain $G = \mathbb{Z}_7 : \mathbb{Z}_6$, or $\mathbb{Z}_7 : \mathbb{Z}_3$.

Denote by T the socle $\text{soc}(X)$. Assume first that $G = \mathbb{Z}_7 : \mathbb{Z}_6$. Since $\mathbb{Z}_7 : \mathbb{Z}_3$ is maximal in T , we have $X = \text{PGL}(2, 7)$. It follows that $X_u \cong \text{D}_8$. Assume now that $G = \mathbb{Z}_7 : \mathbb{Z}_3$. Furthermore, assume that $X = \text{PSL}(2, 7)$. Then Γ is a connected tetravalent X -edge-transitive Cayley graph, and $X_u \cong \text{D}_8$ is a Sylow 2-subgroup of X . Choose $v \in \Gamma(u)$. Then $|X_u : X_{uv}| = 2$ or 4. Since Γ is vertex transitive graph, we write Γ as coset graph $\text{Cos}(X, H, H\{x, x^{-1}\}H)$, where $H = X_u \cong \text{D}_8$, and $x \in X$ is such that $\langle H, x \rangle = X$; in particular, $x \notin H$.

Suppose that $|X_u : X_{uv}| = 4$. Then Γ is X -arc-transitive. By Lemma 2.1, we choose x such that $(u, v)^x = (v, u)$, resulting $x \in \mathbf{N}_X(X_{uv}) \cong \mathbf{D}_8$. In particular, $\mathbf{N}_X(X_{uv}) \neq X_u$. Then $|\mathbf{N}_{X_u}(X_{uv})| = 4$. Hence $\mathbf{N}_{X_u}(X_{uv})$ is normal in both $\mathbf{N}_X(X_{uv})$ and X_u , and so $\mathbf{N}_{X_u}(X_{uv}) \leq \langle X_u, \mathbf{N}_X(X_{uv}) \rangle$. Checking the subgroups of $\mathrm{PSL}(2, 7)$, we obtain that $\langle X_u, \mathbf{N}_X(X_{uv}) \rangle \cong \mathbf{S}_4$, which contradicts the fact $\langle X_u, x \rangle = X$.

Suppose that $|X_u : X_{uv}| = 2$. Then $|X_{uv}| = 4$, and hence $X_{uv} \leq M = \langle X_u, X_v \rangle$, so $M \cong \mathbf{S}_4$. By Lemma 2.1, we may choose x such that $u^x = v$. It is clear that X_u and X_v are two Sylow 2-subgroup of M , there exists some $y \in M$ such that $X_u^y = X_v = X_u^x$. Hence $xy^{-1} \in \mathbf{N}_X(X_u) = X_u$, so $\langle X_u, x \rangle \leq \langle X_u, xy^{-1}, y \rangle \leq M$, again a contradiction. Thus $X = \mathrm{PGL}(2, 7)$. \square

Lemma 5.2 tells us that if X is insoluble and $R(X) = 1$, then X is almost simple. The next two lemmas determine Γ for the case where X is almost simple.

Lemma 5.4. *If X is almost simple, then, for $u \in V\Gamma$, the triple (X, G, X_u) is one of the triples listed in Table 2.*

Proof. By Frattini argument, we have that $X = GX_u$. Since Γ is a connected Cayley graph of valency 4, we obtain that X_u is a $\{2, 3\}$ -group. It follows that X is decomposed as a product of two solvable subgroups. Denote by T the socle $\mathrm{soc}(X)$. By [12], we conclude that T appears in Lemma 5.2. In what follows, we process our analysis by two cases.

Case 1: Assume that Γ is $(X, 2)$ -arc-transitive. By Lemma 2.2, $|X_u|$ divides $2^4 3^6$.

Assume first that $T = \mathrm{PSL}(2, q)$ with $q > 3$. If $q = 5$, then $G = \mathbb{Z}_5$ and $X_u = \mathbf{A}_4$ or \mathbf{S}_4 , a contradiction. If $q = 7$, by the Atlas [4], we conclude that $X = \mathrm{PGL}(2, 7)$, $G = \mathbf{D}_{14}$, and $X_u = \mathbf{S}_4$. Suppose $q = 11$. Checking the subgroups of $\mathrm{PGL}(2, 11)$ in the Atlas [4], we know that $X = \mathrm{PSL}(2, 11) \cdot \mathcal{O}$, $G = \mathbb{Z}_{11} : (\mathbb{Z}_5 \times \mathcal{O}_1)$, and $X_u = \mathbf{A}_4 \cdot \mathcal{O}_2$, where $\mathcal{O} = \mathbb{Z}_2$, and $\mathcal{O}_1 \mathcal{O}_2 = \mathcal{O}$, (refer to [18, Theorem 1.5]). Suppose that $q = 23$. By [17, Theorem 1.1], $X = \mathrm{PSL}(2, 23)$, $G = \mathbb{Z}_{23} : \mathbb{Z}_{11}$ and $X_u = \mathbf{S}_4$.

In what follows, we assume that $q \neq 4, 5, 7, 11, 23$. Then, by [20, Proposition 4.1], interchanging G and X_u if necessary, $G \cap T \leq \mathbf{D}_{2(q+1)/d}$ and $[q] \leq T \cap X_u \leq [q] : [\frac{q-1}{d}]$ where $d = (2, q-1)$. Let $T_u = T \cap X_u$. Assume that $G \cap T \leq \mathbf{D}_{2(q+1)/d}$, and $[q] \leq T_u \leq [q] : [\frac{q-1}{d}]$. Then $\frac{q(q-1)}{2}$ divides $|T : T \cap G|$. Since $|T : T \cap G| = |TG : G|$ divides $|X_u|$, we conclude that $\frac{q(q-1)}{2} \mid 2^4 3^6$. By easy calculations, $q = 9$. It follows that $T_u^{\Gamma(u)} \cong \mathbb{Z}_3$ or \mathbf{S}_3 . However, $T_u^{\Gamma(u)} \leq X_u^{\Gamma(u)} \leq \mathbf{S}_4$, and so $X_u^{\Gamma(u)} \cong \mathbf{S}_3$, which is a contradiction because $X_u^{\Gamma(u)}$ is 2-transitive on $\Gamma(u)$. Thus $[q] \leq T \cap G \leq [q] : [\frac{q-1}{d}]$ and $T_u \leq \mathbf{D}_{2(q+1)/d}$. Then $T_u^{\Gamma(u)} \cong \mathbf{S}_3$ or \mathbb{Z}_3 , and so $X_u^{\Gamma(u)} \cong \mathbf{S}_3$, again a contradiction.

Assume now that $T = \mathrm{PSL}(3, q)$ ($q < 9$), $\mathrm{PSp}(4, 3)$, $\mathrm{PSL}(4, 2)$, $\mathrm{PSU}(3, 8)$ or \mathbf{M}_{11} . It is clear that $T \cap G \neq 1$. Then $W \leq T$. Suppose that $T = \mathrm{PSp}(4, 3)$. Using [20, Proposition 4.1], $T \cap G = \mathbb{Z}_2^4 : \mathbb{Z}_5$ and $T \cap X_u = 3_+^{1+2} : 2\mathbf{A}_4$. For that case, it is clear that $G \cap X_u \neq 1$, a contradiction. Suppose that $T = \mathbf{M}_{11}$. Then $X = \mathbf{M}_{11}$. Again by [20, Proposition 4.1], $G = \mathbb{Z}_{11} : \mathbb{Z}_5$, and $X_u = \mathbf{M}_9 \cdot 2$, again a contradiction, refer to Lemma 2.2. Suppose that $T = \mathrm{PSL}(3, 4)$. By Atlas [4], we conclude that $35 \nmid |T \cap G|$. It implies that 5 or 7 divides $|X_u|$, which is impossible. Similarly, T does not equal $\mathrm{PSL}(3, q)$ with $5 \leq q \leq 8$, $\mathrm{PSL}(4, 2)$, or $\mathrm{PSU}(3, 8)$. If $T = \mathrm{PSL}(3, 3)$, we obtain that $X = \mathrm{PSL}(3, 3) : \mathbb{Z}_2$, $G = \mathbf{D}_{26}$ and $X_u = \mathbb{Z}_3^2 : \mathrm{GL}(2, 3)$, refer to [20, Proposition 4.1].

Case 2: Assume that Γ is not $(X, 2)$ -arc-transitive. By Case 1, we only need deal with the case where $T = \text{PSL}(2, q)$ with $q > 3$. Since Γ is not $(X, 2)$ -arc-transitive, X_u is a 2-group. Since $X = GX_u$ and $G \cap X_u = 1$, G contains a $2'$ -Hall subgroup of X . Then $G \cap T$ contains a $2'$ -Hall subgroup of T . By Lemma 2.5, we have that $T = \text{PSL}(2, q)$, $T \cap G = \mathbb{Z}_q : \mathbb{Z}_{\frac{q-1}{2}}$, and $T_u = D_{q+1}$, where $q = 2^e - 1$ is a prime (see [20, Proposition 4.1]). Suppose that $e = 3$. Then $q = 7$. By Lemma 5.3, the statement follows. In what follows, we assume that $e \geq 5$.

Note that $\mathbb{Z}_q : \mathbb{Z}_{\frac{q-1}{2}}$ is maximal in T . By [19, Theorem 1.1], we conclude that $G = \mathbb{Z}_q : \mathbb{Z}_{q-1}$, and hence $X = \text{PGL}(2, q)$, and $X_u = T_u = D_{q+1}$. Let $v \in \Gamma(u)$. By Lemma 2.1, T_{uv} has index 2 or 4 in both T_u and T_v . Since $e \geq 5$, T_{uv} contains a subgroup $C \cong \mathbb{Z}_4$. It is easily shown that C is normal in both T_u and T_v , and so $C \triangleleft L := \langle T_u, T_v \rangle$. In view of [28, p.417], each Sylow 2-subgroup of T is maximal in T . Note that T_u is a Sylow 2-subgroup of T , it follows that $L = T_u = T_v$. By the connectedness of Γ , we obtain L fixes each vertex of Γ , which is impossible. This completes the proof of Lemma 5.4. \square

By [17, Theorem 1.1], we have the following lemma.

Lemma 5.5. *Let X be in Table 2, and let $T = \text{soc}(X)$. Then we have:*

- (1) *if $T = \text{PSL}(3, 3)$, then Γ is isomorphic to the graph given in Example 3.13;*
- (2) *if $T = \text{PSL}(2, 7)$, then Γ is isomorphic to a graph given in Examples 3.13, 3.14 and 3.15;*
- (3) *if $T = \text{PSL}(2, 23)$, then Γ is isomorphic to the graph given in Example 3.16;*
- (4) *if $T = \text{PSL}(2, 11)$, then Γ is isomorphic to a graph given in Examples 3.16-3.17.*

We now begin with treating the case where $R(X) \neq 1$.

Lemma 5.6. *If $R(X) \cap G = 1$, then $G = \mathbb{Z}_{11} : \mathbb{Z}_{10}$ and $X = \text{PGL}(2, 11) \times \mathbb{Z}_2$.*

Proof. Let $\overline{X}_1 = X_1 R(X) / R(X)$, $\overline{G} = GR(X) / R(X)$, and $\overline{X} = X / R(X)$. Since $X = GX_1$, we conclude that $\overline{X} = \overline{G} \overline{X}_1$. Since $R(X) \cap G = 1$, implying that $\overline{G} \cong G$ is a Frobenius group. Since \overline{X} is insoluble, Γ is a cover of $\Gamma_{R(X)}$, refer to Lemma 2.3. Let B be a vertex of $\Gamma_{R(X)}$, where $1 \in B$. By Frattini argument, $\overline{X} = \overline{G} \overline{X}_B$. Clearly, $\overline{X}_1 \leq \overline{X}_B$, and $\overline{G} \cap \overline{X}_B \neq 1$.

Assume that \overline{X} is not almost simple. Let \overline{N} be a minimal normal subgroup of \overline{X} . Arguing similarly as Lemma 5.1, we obtain that $\mathbf{C}_{\overline{X}}(\overline{N}) = 1$. Suppose that $\text{soc}(\overline{X}) = A_5 \times A_5$. By the definition of G , we conclude that $\overline{G} \cong \mathbb{Z}_5^2 : \mathbb{Z}_8$. For this case, $\overline{X} = ((A_5 \times A_5) : \mathbb{Z}_2) : \mathbb{Z}_2$ is a subgroup of $S_5 \wr S_2$, and $\overline{X}_B \cong S_3 \times S_4$. It implies that $\overline{G} \cap \overline{X}_B \cong \mathbb{Z}_2$. By MAGMA [2], there are two elements \overline{G} (up to conjugate) in \overline{X} , and there is an element \overline{X}_B (up to conjugate) in \overline{X} such that their intersections are isomorphic to \mathbb{Z}_2 . Choose $\overline{B} \in \Gamma(B)$. By Lemma 2.1, write $\Gamma_{R(X)} = \text{Cos}(\overline{X}, \overline{X}_B, \overline{X}_B o \overline{X}_B)$, where $o \in \mathbf{N}_{\overline{X}}(\overline{X}_{B\overline{B}}) \setminus \overline{X}_B$ and $o^2 \in \overline{X}_{B\overline{B}}$. Since Γ is \overline{X} -arc-transitive graph, we conclude $|\overline{X}_B : \overline{X}_{B\overline{B}}| = 4$, and hence $|\overline{X}_{B\overline{B}}| = 36$. Again by MAGMA [2], for each choice of \overline{G} and \overline{X}_B , there is no $o \in \mathbf{N}_{\overline{X}}(\overline{X}_{B\overline{B}})$ such that $\langle \overline{X}_B, o \rangle = \overline{X}$, namely, $\Gamma_{R(X)}$ is not connected. For other cases, arguing similarly as Lemma 5.2, we exclude these possibilities. Thus \overline{X} is almost simple.

Let $\overline{T} = \text{soc}(\overline{X})$. By [12], we obtain that \overline{T} is one of the following:

$\text{PSL}(2, q) (q > 3)$, $\text{PSL}(3, q) (q < 9)$, $\text{PSL}(4, 2)$, $\text{PSp}(4, 3)$, $\text{PSU}(3, 8)$, or M_{11} .

Suppose first that $\overline{T} = \text{PSL}(2, q)$ where $q = 5, 7, 11, 23$. If $q = 5$, the only possibility is that $\overline{G} \cong \mathbb{Z}_5 : \mathbb{Z}_4$, refer to [17, Theorem 1.1]. For this case, Γ is a Cayley graph of order 20, by [24, Theorem 5.3], G is normal in X , which is a contradiction. If $q = 7, 11$ or 23 , then $|G|$ is square-free, refer to Atlas [4]. Again by [17, Theorem 1.1], $X = \text{PGL}(2, 11) \times \mathbb{Z}_2$, $G = \mathbb{Z}_{11} : \mathbb{Z}_{10}$, $X_1 = S_4$ and Γ is isomorphic to the graph given in Example 3.17. Arguing similarly as Lemma 5.4 with $\overline{X} = \overline{G} \overline{X}_B$ in the place $X = GX_1$, we can exclude other cases. This completes the proof. \square

Lemma 5.7. *If $R(X) \cap G \neq 1$, then we have:*

- (a) $G \cong \mathbb{Z}_p^4 : \mathbb{Z}_5$, $X = W.\overline{X}$, and $\Gamma_W = \mathbf{K}_5$, where $\text{soc}(\overline{X}) \cong A_5$;
- (b) $G \cong \mathbb{Z}_p^4 : \mathbb{Z}_{10}$, $X = W.(\overline{X} \times \mathbb{Z}_2)$, and $\Gamma_W = \mathbf{K}_{5,5} - 5\mathbf{K}_2$, where $\text{soc}(\overline{X}) \cong A_5$.

Proof. Let $R = R(X) \cap G$. Assume that $R(X) \cap G \neq 1$. From the minimality of W in G , it follows that $R \geq W$. Since $X/R(X)$ is insoluble, by Lemma 2.3, Γ is a normal cover of $\Gamma_{R(X)}$. Hence $GR(X)/R(X) \leq \text{Aut}\Gamma_{R(X)}$.

We claim that $R = R(X)$. Let $\overline{H} = HR(X)/R(X)$. Since Γ is a Cayley graph of G , implying that $\Gamma_{R(X)}$ is a connected Cayley graph of \overline{H} . It follows that $|R(X)||\overline{H}| = |G|$, and hence $|R(X)| = |W||R(X) \cap H|$. So $R(X) \leq G$, and then $R = R(X)$, as claimed. Furthermore, it implies that W is a normal subgroup of X . By [1, Theorem 1.2], we obtain that either $\Gamma_W \cong \mathbf{K}_5$ and $H \cong \mathbb{Z}_5$ or $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$ and $H \cong \mathbb{Z}_{10}$. In the former case, we easily obtain $\text{Aut}\Gamma_W \cong S_5$, and in the latter case, $\text{Aut}\Gamma_W \cong S_5 \times \mathbb{Z}_2$.

Let $\overline{X} = X/W$. Since Γ is a cover of Γ_W , it follows that \overline{X} is a subgroup of $\text{Aut}\Gamma_W$. Assume first that $\Gamma_W \cong \mathbf{K}_5$. Notice that \overline{X} is insoluble, we conclude that $\text{soc}(\overline{X}) \cong A_5$. Assume now that $\Gamma_W \cong \mathbf{K}_{5,5} - 5\mathbf{K}_2$. Since $H \cong \mathbb{Z}_{10}$, and X is insoluble, we obtain $\overline{X} \cong L \times \mathbb{Z}_2$ where $\text{soc}(L) \cong A_5$.

Recall that H is irreducible on W , we conclude that $d = 4$, refer to [33, Lemma 3.1]. Hence $G \cong \mathbb{Z}_p^4 : \mathbb{Z}_5$ or $\mathbb{Z}_p^4 : \mathbb{Z}_{10}$. \square

The assertion of Theorem 1.2 follows from Lemmas 5.2-5.7.

6. HALF-TRANSITIVE GRAPHS

In the last section, we aim to prove Theorem 1.3.

Let p be an odd prime, and $d > 1$ be an odd integer. Let n be a primitive divisor of $p^d - 1$, and let

$$G = W : \langle h \rangle = \mathbb{Z}_p^d : \mathbb{Z}_n < \text{AGL}(1, p^d).$$

Construction 6.1. Let i be coprime to n such that $1 \leq i \leq n - 1$, and let $a \in W \setminus \{1\}$. Let

$$\begin{cases} S_i = \{ah^i, a^{-1}h^i, (ah^i)^{-1}, (a^{-1}h^i)^{-1}\}, \\ \Gamma_i = \text{Cay}(G, S_i). \end{cases}$$

Proof of Theorem 1.3: Let $X = \text{Aut}\Gamma$. Let $\Gamma = \text{Cay}(G, S)$ be connected, edge-transitive and of valency 4. Note that $\langle h \rangle$ is primitive on W , $d > 1$ is odd, and p is an odd prime. By Theorem 1.1 and Theorem 1.2, we obtain that G is normal in X . In view of [10, Lemma 2.1], we have $X = G : \text{Aut}(G, S)$.

By Lemma 2.4, we have that $X_1 = \text{Aut}(G, S) \leq D_8$. By [8, Proposition 12.10], $\text{Aut}(G) \cong \mathbb{Z}_p^d : (\mathbb{Z}_{p^d-1} : \mathbb{Z}_d)$. Since d and p are odd, $\text{Aut}(G)$ has a cyclic Sylow 2-subgroup.

It follows that $X_1 = \langle \sigma \rangle \cong \mathbb{Z}_4$ or \mathbb{Z}_2 . Thus σ fixes an element of G of order n , say $f \in G$ such that $o(f) = n$ and $f^\sigma = f$. Then $G = W:\langle f \rangle$, and $X = G:\langle \sigma \rangle = (W:\langle f \rangle):\langle \sigma \rangle$. Moreover, it implies that all involutions of $\text{Aut}(G)$ are conjugate. Recall that G is a Frobenius group, every involution of $\text{Aut}(G)$ inverts all elements of W .

Since Γ is connected, $\langle S \rangle = G$ and $\text{Aut}(G, S)$ is faithful on S . Assume that S contains an involution. Recall that Γ is X -edge-transitive, we conclude that S consists of involutions. By the proof of Lemma 2.4, $G \cong D_{2p}$, against our assumption. Hence S does not contain an involution. For that case, we may write $S = \{x, x^{-1}, y, y^{-1}\}$ such that either $o(\sigma) = 2$ and $(x, y)^\sigma = (y, x)$, or $o(\sigma) = 4$ and $(x, y)^\sigma = (y, x^{-1})$, refer to [27, Proposition 1]. Now $x = af^i$, where $a \in W$ and i is an integer. Suppose that $o(\sigma) = 4$. Then $y = x^\sigma = (af^i)^\sigma = a^\sigma f^i$, and $a'f^{-i} = f^{-i}a^{-1} = (af^i)^{-1} = x^{-1} = x^{\sigma^2} = a^{\sigma^2}f^i = a^{-1}f^i$. It follows that $f^{2i} = 1$, and hence f^i has order 1 or 2. If $f^i = 1$, then $x = a$, and $y = x^\sigma = a^\sigma$, belonging to W , and so $\langle S \rangle \leq W < G$, which is a contradiction. Thus f^i has order 2. Note that f^i inverts each element of W , we conclude that x has order 2, again a contradiction. Thus σ is an involution, and so $(x, y)^\sigma = (y, x)$, $x = af^i$, and $y = x^\sigma = a^\sigma f^i = a^{-1}f^i$. In particular, Γ is not arc-transitive, and $S = \{af^i, a^{-1}f^i, (af^i)^{-1}, (a^{-1}f^i)^{-1}\}$.

Since $f \in G$ has order n , it follows from Hall's theorem that there exists $b \in W$ such that $h^b \in \langle f \rangle$. So $f^{b^{-1}} = h^r$ for some r . Let $\tau = \sigma^{b^{-1}}$. Then τ centralizes $\langle h \rangle$, and $X = G:\langle \tau \rangle$. Moreover, $S^{b^{-1}} = \{ah^{ir}, a^{-1}h^{ir}, (ah^{ir})^{-1}, (a^{-1}h^{ir})^{-1}\}$. Let $ir \equiv j \pmod{n}$, and $1 \leq j \leq n-1$. Then $S_j := \{ah^j, a^{-1}h^j, (ah^j)^{-1}, (a^{-1}h^j)^{-1}\}$. Note that $\Gamma \cong \text{Cay}(G, S_j)$ is connected. By [19, Lemma 6.2], $(j, n) = 1$, $\Gamma_i \cong \Gamma_{n-i}$, and if $p^k i \equiv \pm j \pmod{n}$ for some k , then $\Gamma_i \cong \Gamma_j$. This completes the proof of Theorem 1.3. \square

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